Chapter 1. Linear Spaces and Operators

1.1. Introduction

Note. Traditionally the sequence of analysis classes is as follows:

1. Senior-level analysis (Analysis 1 [MATH 4217/5217] and Analysis 2 [MATH 4227/5227]),

2. Graduate-level analysis (Real Analysis 1 [MATH 5210]) and Real Analysis 2 [MATH 5220]), and

3. Functional Analysis.

Senior-level analysis deals basically with the topics of calculus, but in a rigorous axiomatic way. The real numbers are defined (as a complete ordered field), and limits, sequences, continuous functions, differentiability, Riemann integration, and sequences and series of functions are explored. In graduate-level analysis, Riemann integration is generalized with the introduction of Lebesgue measure and Lebesgue integration. Other topics may include metric spaces, topological spaces, Banach spaces, Hilbert spaces, and additional measure theory (such as signed measure, product measure, and other abstract measures). Functional analysis involves the study of Banach spaces, Hilbert spaces, and operators on these spaces.

Note. With Analysis, Real Analysis, and Functional Analysis classes as background, one would be prepared for a graduate class in partial differential equations or probability theory. Neither of these classes are offered at ETSU, but I do have online notes for a graduate class in Measure Theory Based Probability.

Note. In the past, several topics from functional analysis have been included in the graduate-level Real Analysis classes, due to the facts that we had no functional analysis class and that we only have a master's program. It is traditional to take the analysis classes in the order given above, with graduate analysis as a prerequisite for functional analysis. However, our text *A First Course in Functional Analysis* by S. David Promislow (John Wiley and Sons, 2008), allows us to address several topics from functional analysis without a prior knowledge of measure theory or Lebesgue integration. Basic facts about measurable functions and integrals are summarized in the text without detailed proofs. For notes on a more traditional version of functional analysis, see my online notes for Functional Analysis based on M. Reed and B. Simon's *Functional Analysis I*, Revised and Enlarged Edition (Academic Press, 1980).

Note. Informally put,

(1) a *function* maps numbers to numbers,

(2) an *operator* maps functions to functions, and

(3) a *functional* maps functions to numbers.

The text briefly describes this course as "the study of *bounded linear operators on normed linear spaces.*" As a consequence, we start with a study of linear spaces (which, you will see, are really just vector spaces) and linear operators (examples of which are matrices in the setting of finite dimensional vector spaces).