1.2. Linear Spaces

Note. As in vector spaces, we need a field of scalars. We will usually use the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or a field $\mathbb{F}$.

Note. For a complex number $z \in \mathbb{C}$, we denote $z = x+iy$ where $x, y \in \mathbb{R}$ and define the *real part* of z as $\mathrm{Re}(z) = x$ and the *imaginary part* of z $\mathrm{Im}(z) = y$. The *conjugate* of z is $\overline{z} = x - iy$. The *modulus* or *absolute value* of z is $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$. We can verify using series (and trusting that the series converge absolutely) that $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$. This allows us to represent complex numbers in *polar form* $z = |z|e^{i\theta}$ where $\tan\theta = y/x$.

Definition. A *linear space* over the field $\mathbb{F}$ is a set X endowed with the operations of addition $+$ and scalar multiplication of elements of X by elements of $\mathbb{F}$ which for all $x, y \in X$ and for all $\alpha \in \mathbb{F}$ we have $x + y \in X$ and $\alpha x \in X$, and:

(1) $x + y = y + x$ (Commutivity of $+$).

(2) $(x + y) + z = x + (y + z)$ (Associativity of $+$).

(3) There is $0 \in X$ such that $x + 0 = x$.

(4) For each $x \in X$ there is $-x \in X$ such that $x + (-x) = 0$.

(5) $1x = x$.

(6) $\alpha(\beta x) = (\alpha\beta)x$.

(7) $(\alpha+\beta)x = \alpha x+\beta x$ (Distribution of Scalar Multiplication over Scalar Addition).

(8) $\alpha(x+y) = \alpha x+\beta y$ (Distribution of Scalar Multiplication over Vector Addition).

We denote this linear space simply as X.

Note. These spaces are called "linear spaces" because they involve linear combinations of elements of X. You may have noticed that the definition of "linear space" is equivalent to the definition of "vector space"; see my online notes for Linear Algebra on Section 3.1. Vector Spaces (see Definition 3.1). Therefore, examples of linear spaces are $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{F}^n$ (where $\mathbb{F}$ is a field).

Definition. If A and B are subsets of X, and $\alpha \in \mathbb{F}$ then we define $A + B = \{a + b \mid a \in A, b \in B\}$ and $\alpha A = \{\alpha a \mid a \in A\}$.

Definition. Let $Y \neq \varnothing$, $Y \subset X$ where X is a linear space. If Y itself is a linear space then Y is a *subspace* of X.

Definition. Given a set B in a linear space, a *finite linear combination* of elements of B is an element x of the form $x = \sum_{i=1}^{n} \alpha_i b_i$ where $\alpha_i \in \mathbb{F}$ and $b_i \in B$ for each $i = 1, 2, \ldots, n$. The $\alpha_i \in \mathbb{F}$ are the *coefficients* of the linear combination.

Note. In linear algebra, you see that $Y \subset X$ is a subspace of X if Y is closed under (1) vector addition, and (2) scalar multiplication. See my online Linear Algebra notes on Section 3.2. Basic Concepts of Vector Spaces (see "Theorem 3.2. Test for Subspace").

Definition. Given a set B in a linear space, the *span* of B, denoted $\mathrm{span}(B)$, is the set of all finite linear combinations of elements of B.

Definition. A set B is a *basis* for linear space X if every element in X can be written uniquely as a finite linear combination of elements of B with nonzero coefficients.

Definition. A linear space is *finite dimensional* if it has a basis with a finite number of elements.

Note. It can be shown in linear algebra that in a finite dimensional linear space, all bases have the same number of elements (see "Corollary 3.2.A. Invariance of Dimension for Finitely Generated Spaces" in Section 3.2. Basic Concepts of Vector Spaces). This common number is called the *dimension* of the space. In fact, in an infinite dimensional linear space, it can be shown that all bases have the same cardinality. Though it is not part of the definition of "linear space," it can be shown that every linear space *has* a basis. We will show this later in Supplement. Groups, Fields, and Vector Spaces.

Definition. A set B in a linear space is said to be *linearly independent* if no element in B can be written as a finite linear combination of the others.

Note. Equivalent to the definition of "set B is linearly independent" is that any finite linear combination of elements of B which yield 0 must have only 0 coefficients. (Notice our casual notation here! There is a big distinction between the *scalar* 0 and the *vector* 0. In linear algebra, we notationally distinguish between vectors and scalars, usually be putting little arrows over the vector, $\vec{0}$, or by bold facing vectors, $\mathbf{0}$. In our text, we depend on the context to distinguish between scalars and vectors.)

Note. We could have defined a basis of a linear space as a linearly independent spanning set, as is often done in linear algebra (see Definition 3.6 in Section 3.2. Basic Concepts of Vector Spaces).

Note. The main form of linear space of interest to us will be a linear space of functions. One elementary example is

$$X = \{a\cos(x) + b\sin(x) \mid a, b \in \mathbb{R}\}.$$

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