1.3. Linear Operators

Note. In this section we define one of the most fundamental objects of functional analysis: The linear operator. The rest of this course is devoted to studying properties of and classifying linear operators on linear spaces.

Definition. If X and Z are two linear spaces over the same scalar field, a *linear operator* from X to Z is a rule T that associates with each $x \in X$ a unique element $Tx \in Z$ such that for all $x, y \in X$ and $\alpha \in \mathbb{F}$ we have $T(x+y) = T(x) + T(y)$ and $T(\alpha x) = \alpha T(x)$. Linear operators are also referred to as *linear transformations* or *linear mappings*.

Note. We can combine the two above properties and define T as linear if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in X$ and for all $\alpha, \beta \in \mathbb{F}$.

Note. In linear algebra, you see that a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$ is equivalent to an $m \times n$ matrix (recall that the elements of $\mathbb{R}^n$ are n-tuples of real numbers). See my online Linear Algebra (MATH 2010) notes on Section 3.4. Linear Transformations (see "Theorem 3.10. Matrix Representations of Linear Transformations").

Definition. The *null space* (or *kernel*) of linear operator $T : X \to Z$, denoted $N(T)$, is the set of all $x \in X$ such that $T(x) = 0$. The *range* of T (also called the *image* of X under T), denoted $R(T)$, is the set of all $z \in Z$ such that $Tx = z$ for some $x \in X$.

Definition. A linear operator $T : X \to Z$ is *surjective* (or *onto*) if $R(T) = Z$. T is *injective* (or *one-to-one*) if $Tx = Ty$ implies $x = y$.

Note. In Linear Algebra (MATH 2010) we show that a linear transformation T between vector spaces is injective (one-to-one) if and only if the kernel of T consists only of the 0 vector (see "Corollary 3.4.A. One-to-One and Kernel" in Section 3.4. Linear Transformations). We present the same result here in our notation.

Theorem 1.3.A. Linear operator $T : X \to Z$ is injective if and only if $N(T) = 0$.

Note. With $T : X \to Z$ a linear operator, we can consider the equation $Tx = z$ where $x \in X$ and $z \in Z$. A solution $x \in X$ exists for all $z \in Z$ if and only if $R(T) = Z$. Now if a solution $x \in X$ exists for each $z \in Z$, then it is unique if and only if $N(T) = 0$ since $Tx = Ty$ if and only if $x - y \in N(T)$.

Definition. If $N(T) = 0$, then we define $T^{-1} : R(T) \to X$ as $T^{-1}z = x$ where $Tx = z$.

Theorem 1.3.B. Let $T : X \to Z$ be linear with $N(T) = 0$. Then $T^{-1} : R(T) \to X$ is linear.

Note. Let X and Z be linear spaces. If T_1 and T_2 are linear transformations from X to Z, then for scalars $\alpha, \beta \in \mathbb{F}$ we can define the transformation $\alpha T_1 + \beta T_2 : X \to Z$ as $(\alpha T_1 + \beta T_2)(x) = \alpha T_1(x) + \beta T_2(x)$. Then since for $\gamma, \delta \in \mathbb{F}$ and $x, y \in X$ we have:

$$\begin{aligned}
(\alpha T_1 + \beta T_2)(\gamma x_1 + \delta x_2) &= \alpha T_1(\gamma x_1 + \delta x_2) + \beta T_2(\gamma x_1 + \delta x_2) \\
&\quad \text{by definition of } \alpha T_1 + \beta T_2 \\
&= \alpha(T_1(\gamma x_1) + T_1(\delta x_2)) + \beta(T_2(\gamma x_1) + T_2(\delta x_2)) \\
&\quad \text{since } T_1 \text{ and } T_2 \text{ are linear} \\
&= \alpha T_1(\gamma x_1) + \alpha T_1(\delta x_2) + \beta T_2(\gamma x_1) + \beta T_2(\delta x_2) \\
&\quad \text{by distribution of scalar multiplication} \\
&\quad \text{over vector addition} \\
&= \alpha\gamma T_1(x_1) + \alpha\delta T_1(x_2) + \beta\gamma T_2(x_1) + \beta\delta T_2(x_2) \\
&\quad \text{since } T_1 \text{ and } T_2 \text{ are linear} \\
&= \gamma(\alpha T_1(x_1) + \beta T_2(x_1)) + \delta(\alpha T_1(x_2) + \beta T_2(x_2)) \\
&\quad \text{by distribution of scalar multiplication} \\
&\quad \text{over vector addition} \\
&= \gamma(\alpha T_1 + \beta T_2)(x_1) + \delta(\alpha T_1 + \beta T_2)(x_2) \\
&\quad \text{by definition of } \alpha T_1 + \beta T_2.
\end{aligned}$$

So we can create a linear space out of the linear operators from X to Z.

Definition. For linear spaces X and Z, the linear space of all linear operators from X to Z is denoted $\mathcal{L}(X, Z)$. If $X = Z$, we denote this as $\mathcal{L}(X)$.

Note. If $X = \mathbb{R}^n$ and $Z = \mathbb{R}^m$, then $\mathcal{L}(X, Z)$ is the set of all $m \times n$ matrices. We show this in Linear Algebra (MATH 2010); see "Corollary 2.3.A. Standard Matrix Representation of Linear Transformations" in Section 2.3. Linear Transformations of Euclidean Spaces.

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