Chapter 2. Normed Linear Spaces: The Basics

2.1. Metric Spaces

Note. A metric is simply a way to measure distance. If we can measure distance, then we can reproduce many of the properties of $\mathbb{R}$: open/closed sets, limits, continuity, boundedness, connectedness, and compactness. These ideas are covered in Complex Analysis 1 (MATH 5510) (see my online notes on Chapter II. Metric Spaces and the Topology of $\mathbb{C}$) and possibly in the Real Analysis sequence (MATH 5210-5220) (see my online notes on II. Abstract Spaces: General Properties).

Definition. Let X be a set and $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ such that for all $x, y, z \in X$:

(i) $d(x, y) \le d(x, z) + d(z, y)$ (Triangle Inequality).

(ii) $d(x, y) = d(y, x)$ (Symmetry).

(iii) $d(x, y) = 0$ if and only if $x = y$.

The function d is a *metric* and the pair (X, d) is a *metric space*.

Example. With $X = \mathbb{R}$ and $d(x, y) = |x - y|$, we have the usual metric space $(\mathbb{R}, d)$ where we do senior level analysis. More generally, with $X = \mathbb{R}^n$ and

$$d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$$

is also a metric space.

Definition. For set $A \subset X$ in metric space (X, d) and point $x \in X$, define the *distance* from x to A as $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.

Note. We can have $d(x, A) = 0$, yet have $x \notin A$. Consider, for example, $x = \{0\}$ and $A = (0, 1) \subset \mathbb{R}$.

Definition. A sequence $\{x_n\}$ in a metric space (X, d) *converges* to a point $x \in X$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x, x_n) < \varepsilon$. x is the *limit* of the sequence and we write $x = \lim x_n$ or $x_n \to x$.

Definition. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A function $f : X_1 \to X_2$ is *continuous* at point $p \in X_1$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that $d_1(p, q) < \delta$ implies that $d_2(f(p), f(q)) < \varepsilon$.

Theorem 2.1.A. A function $f : X_1 \to X_2$ is continuous at point $p \in X_1$ if and only if for each sequence $\{x_n\} \subset X_1$ with $x_n \to p$, we have $\lim f(x_n) = f(p)$.

Note. The proof of Theorem 2.1.A is to be given in Exercise 2.1.

Definition. Let (X, d) be a metric space and $A \subset X$. The *diameter* of A is

$$\operatorname{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

Set A is *bounded* if $\operatorname{diam}(A) < \infty$.

Note. Notice the similarity in the definition of diameter of a set and diameter of a graph; see my online notes for Graph Theory 1 (MATH 5340) on Section 3.1. Walks and Connection.

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