2.10. Direct Products and Sums

Note. In this section we briefly introduce two ways to create new normed linear spaces from old ones.

Definition. Let $S$ be a set (called an indexing set) and for each $s \in S$ suppose $X_s$ is a linear space. Consider the set of all functions on $S$ such that $f(s) \in X_s$ for all $s \in S$. (Notice $f : S \to \bigcup_{s \in S} X_s$. Also, since $X_s$ is a linear space, we can take linear combinations of such $f$'s.) The set, itself a linear space, is the product of the spaces, denoted $\prod_{s \in S} X_s$.

Example. If $S = \{1, 2\}$ then the product is the usual $X_1 \times X_2$.

Definition. If $X_s$ is a normed linear space for all $s \in S$, then for $f \in \prod_{s \in S} X_s$ define $\|f\| = \sup\{\|f\| \mid s \in S\}$.

Claim. $\| \cdot \|$ defined on $\prod_{s \in S} X_s$ above is a norm (the “sup norm”) on $X = \{f \in \prod_{s \in S} \mid \|f\| < \infty\}$. This normed linear space is the direct product of the normed linear spaces $X_s$ where $s \in S$. If each $X_s$ is a Banach space, then the direct product is a Banach space.
**Definition.** The natural projection \( \pi_s \) on \( S \) is defined as \( \pi_s(f) = f(s) \) for each \( s \in S \).

**Definition.** The subspace of \( \prod_{s \in S} X_s \) which consists of all functions that take a value of zero, except for finitely many values of \( s \) is the direct sum of \( X_s \) for \( s \in S \).

**Note.** Direct sums of normed linear spaces \( X_s \)'s admit more norms than direct products (as the text claims on page 52). If set \( S \) is finite, then the direct product and direct sum are the same.

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