2.10. Direct Products and Sums

Note. In this section we briefly introduce two ways to create new normed linear spaces from old ones.

Definition. Let S be a set (called an *indexing set*) and for each $s \in S$ suppose X_s is a linear space. Consider the set of all functions on S such that $f(s) \in X_s$ for all $s \in S$. (Notice $f : S \to \cup_{s \in S} X_s$. Also, since X_s is a linear space, we can take linear combinations of such f's.) The set, itself a linear space, is the *product* of the spaces, denoted $\prod_{s \in S} X_s$.

Example. If $S = \{1, 2\}$ then the product is the usual $X_1 \times X_2$.

Definition. If X_s is a normed linear space for all $s \in S$, then for $f \in \prod_{s \in S} X_s$ define $\|f\| = \sup\{\|f(s)\| \mid s \in S\}$.

Theorem 2.10.A. $\| \cdot \|$ defined on $\prod_{s \in S} X_s$ above is a norm (the "sup norm") on $X = \{f \in \prod_{s \in S} X_s \mid \|f\| < \infty\}$. This normed linear space is the *direct product* of the normed linear spaces X_s where $s \in S$. If each X_s is a Banach space, then the direct product is a Banach space.

Definition. The *natural projection* π_s of $f \in \prod_{s \in S} X_s$ to X_s is defined as $\pi_s(f) = f(s)$ for each $s \in S$.

Definition. The subspace of $\prod_{s \in S} X_s$ which consists of all functions that take a value of zero, except for finitely many values of s is the *direct sum* of X_s for $s \in S$.

Note. You may have also seen several of the ideas above in Modern Algebra 1 (MATH 5410) in the setting of groups as direct product, weak direct product, and canonical ("natural") projection; see my online notes on Section I.8. Direct Products and Direct Sums.

Note. Direct sums of normed linear spaces X_s's admit more norms than direct products (as the text claims on page 52). If set S is finite, then the direct product and direct sum are the same.

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