2.11. Schauder Basis

Note. In this brief (two paragraphs) section, we give the definition of two types of bases for normed linear spaces. A much more detailed version of this can be found in Section 5.1. Groups, Fields, and Vector Spaces, supplemental notes based on *Real Analysis with an Introduction to Wavelets and Applications* by Hong, Wang, and Gardner (Elsevier Press, 2005).

Definition. A sequence (x_n) in a normed linear space $(X, \|\cdot\|)$ is a *Schauder basis* if for every $x \in X$ there is a unique collection of scalars $\alpha_n \in \mathbb{F}$ where $x = \sum_{n=1}^{\infty} \alpha_n x_n$.

Note. Since there is a norm on the space, we can discuss limits of sequences and sums of series.

Definition/Note. In Linear Algebra (MATH 2010), we require that every element of a vector space be written as a *finite* linear combination of basis elements. Such a basis is often called a *Hamel basis*. None of this is an issue in Linear Algebra since you primarily study $\mathbb{R}^n$ (or maybe $\mathbb{C}^n$) in sophomore linear algebra. However, it is often impractical to find a Hamel basis. In fact, it requires an equivalent of the Axiom of Choice to show that every vector space has a (Hamel) basis.

Note. Stefan Banach conjectured in 1932 that every separable Banach space has a Schauder basis. However, a counterexample to this conjecture was found in 1973 by Per Enflo (a feat for which he was awarded a live goose by Stanislaw Mazur; see MacTutor History of Mathematics Archive biography of Mazur [accessed 6/3/2021]). Banach's conjecture appears in his 1932 *Theorie des Operations Lineaires*, which is translated into English in *Theory of Linear Operators* (1988), Amsterdam: North Holland Press. Enflo's solution appears in "A Counterexample to the Approximation Problem in Banach Spaces," *Acta Mathematica* 130, 309–317 (1973). You can find a copy of Enflo's paper on the Project Euclid webpage (accessed 6/2/2021). I read into all this that Banach spaces can have difficult and surprising properties. Hilbert spaces are much better behaved.

Note. We will see that every separable Hilbert space (a Hilbert space is a complete inner product space) does have a Schauder basis. In fact, all separable infinite dimensional Hilbert spaces are isomorphic (and isomorphic to ℓ^2).

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