2.12. Fixed Points and Contraction Mappings

Note. In this section we introduce an idea which is powerful both in theory and applications.

Definition. Let $T : X \to X$. If for some $x \in X$ we have that $Tx = x$, then x is a *fixed point* of T.

Note. We briefly enter the realm of metric spaces as opposed to normed linear spaces. Of course, if $(X, \|\cdot\|)$ is a normed linear space, then a metric on X is $d(x, y) = \|x - y\|$.

Definition. A mapping T from a metric space (X, d) to itself is a *contraction mapping* if there exists a $c \in \mathbb{R}$ where $0 < c < 1$ such that for all $x, y \in X$ we have $d(Tx, Ty) \leq cd(x, y)$.

Theorem 2.44. Contraction Mapping Theorem.

A contraction mapping T from a complete metric space to itself has a unique fixed point.

Note. The proof of the Contraction Mapping Theorem hints at its use. We started with *any* element x, iterated the contraction mapping, and this lead us to the fixed point (in the limit). The method of Picard iterates is a technique from differential equations which employs the Contraction Mapping Theorem to find solutions to differential equations.

Corollary 2.45. If T is a mapping from a complete metric space to itself such that T^n is a contraction for some $n \in \mathbb{N}$, then T has a unique fixed point.

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