2.2. Norms

Note. We now introduce a norm on a linear space, use it to induce a metric, and use the metric to address some topological ideas.

Definition. A *norm* on a linear space X is a mapping $\|\cdot\| : X \to [0, \infty)$ such that for all $x, y \in X$ and for all $\alpha \in \mathbb{F}$:

(i) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality).

(ii) $\|\alpha x\| = |\alpha| \|x\|$ (Scalar Property).

(iii) $\|x\| = 0$ implies that $x = 0$.

The pair $(X, \|\cdot\|)$ is a *normed linear space.*

Note. The Scalar Property refers to "$|\alpha|$" and so this requires that field $\mathbb{F}$ itself has a norm on it. We take $\mathbb{F}$ to be either $\mathbb{R}$ or $\mathbb{C}$, so $|\alpha|$ means the absolute value of α (or sometimes "modulus" if $\alpha \in \mathbb{C}$).

Note. A norm $\|\cdot\|$ on a linear space induces a metric d as $d(x, y) = \|x - y\|$.

Examples 2.1 and 2.2. On normed linear spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, we can define the familiar Euclidean norm for $x = (x_1, x_2, \ldots, x_n)$ as

$$\|x\|_2 = \left\{\sum_{k=1}^{n} (x_k)^2\right\}^{1/2} \quad \text{on } \mathbb{R}^n,$$

$$\|x\|_2 = \left\{\sum_{k=1}^{n} x_k \overline{x}_k\right\}^{1/2} \quad \text{on } \mathbb{C}^n.$$

Another norm (called the "ℓ^1 norm") on $\mathbb{R}^n$ and $\mathbb{C}^n$ is

$$\|x\|_1 = \sum_{k=1}^{n} |x_k|.$$

It is easy to see that both of these norms satisfy properties (ii) and (iii) of the definition of norm. It is easy to see that the ℓ^1 norm satisfies the Triangle Inequality (since absolute value and modulus satisfy the Triangle Inequality on $\mathbb{R}$ and $\mathbb{C}$). However, it is more of a chore to see that the Euclidean norm satisfies the Triangle Inequality. This is commonly shown in a Linear Algebra class using the Schwarz Inequality; see my online Linear Algebra notes on Section 1.2. The Norm and Dot Product (see Theorems 1.4 and 1.5).

Note. We can use the Triangle Inequality to establish the Backwards Triangle Inequality:

$$|\|x\| - \|y\|| \le \|x - y\|.$$

Notice $\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$ and so $\|x\| - \|y\| \le \|x - y\|$. Similarly, $\|y\| - \|x\| \le \|y - x\| = \|x - y\|$, and the claim follows.

Theorem 2.3. Continuity of Operations.

Suppose that (x_n) and (y_n) are sequences in a normed linear space, and (α_n) is a sequence in $\mathbb{F}$, and that $x = \lim(x_n)$, $y = \lim(y_n)$, and $\alpha = \lim(\alpha_n)$. Then

(a) $\lim(x_n + y_n) = \lim(x_n) + \lim(y_n) = x + y$.

(b) $\lim(\alpha_n x_n) = \lim(\alpha_n)\lim(x_n) = \alpha x$.

(c) $\lim \|x_n\| = \|x\|$.

Note. The following property is not at all surprising. The surprising thing is that there are certain settings (non-Hausdorff topological spaces) where the result is not true. See my online notes for Analysis 1 (MATH 4217/5217) on Section 3.1. Topology of the Real Numbers (see Bonus 1).

Proposition 2.4. Uniqueness of Limits.

If a sequence (x_n) in a normed linear space converges to both x and y, then $x = y$.

Note. Now that we have a metric defined on linear spaces, we can explore topological properties of linear spaces (open sets, closed sets, compact sets, continuity, etc.).

Definition. For a normed linear space, define the *open r-ball centered at x* as

$$B(x;r) = \{y \in X \mid \|x-y\| < r\}$$

and the *closed r-ball centered at x* as

$$\overline{B}(x;r) = \{y \in X \mid \|x-y\| \leq r\}.$$

In both cases, r is the *radius* of the ball and we require $r > 0$.

Definition. Given any set $A \subseteq X$ and $x \in X$, x is

(i) an *interior point* of A if $B(x;r) \subseteq A$ for some $r > 0$;

(ii) an *exterior point* of A if it is an interior point of $A^c = \{x \in X \mid x \notin A\}$;

(iii) a *boundary point* of A if it is neither interior nor exterior, so any open ball around boundary point x contains both points in A and points in A^c;

(iv) a *limit point* of A if for all $r > 0$ the ball $B(x;r)$ contains a point of A distinct from x.

The set of all boundary points of A is denoted $\partial(A)$.

Note. Now for some very explicitly topological definitions, from which MUCH will follow.

Definition. A set A in a normed linear space is *open* if all points of A are interior points of A. A set A in a normed linear space is *closed* if it contains all of its boundary points.

Note 2.2.A. If A is open then it contains none of its boundary points. Notice that, by definition, a boundary point of A is also a boundary point of A^c. So for open A, A^c is closed. Therefore complements of open sets are closed (this is, often the definition of "closed set"). Vacuously, the empty set $\varnothing$ is both open and closed. Trivially, set X in normed linear space $(X, \|\cdot\|)$ is both open and closed.

Note 2.2.B. As shown in Analysis 1 (MATH 4217/5217), in $\mathbb{R}$ an arbitrary union of open sets is open and a finite intersection of open sets is open. Similarly (by DeMorgan's Laws), an arbitrary intersection of closed sets is closed and a finite union of closed sets is closed. See Theorem 3.3 of my online Analysis 1 notes on Section 3.1. Topology of the Real Numbers. The same properties hold in normed linear spaces. We can also show that a set is closed if and only if it contains all of its limit points (see Corollary 3.6(a) of Section 3.1. Topology of the Real Numbers).

Note. The following definition could be stated in terms of ε's, but the text takes a different route. The following is equivalent to the standard ε/δ definition of continuity in Calculus 1.

Definition. Let $f : X \to Y$ where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are linear spaces. Then f is *continuous at point* $x \in X$ if, given any open ball B' around $f(x)$, there is an open ball B around x such that $f(B) \subseteq B'$.

Definition. Function $f : X \to Y$, where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, is *continuous* if it is continuous at each point $x \in X$.

Note 2.2.C. As in senior level analysis, $f : X \to Y$ is continuous if for all open $V \subseteq Y$, the inverse image $f^{-1}(V)$ is open in X. See Theorem 4.5 of my online notes for Analysis 1 (MATH 4217/5217) on Section 4.1. Limits and Continuity.

Note. The book gives its first ε definition in the following (though it could have done it earlier, since we have a metric in normed linear spaces).

Definition. The function $f : X \to Y$ where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, is *uniformly continuous*, if for all $\varepsilon > 0$ there is a $\delta > 0$ (depending on ε) such that for all $x \in X$ we have $f(B(x;\delta)) \subseteq B(f(x);\varepsilon)$.

Note. Of course, if we choose a single $x_0 \in X$, we see that if f is uniformly continuous on X, then f is continuous at x_0 (that is, uniform continuity implies pointwise continuity). However, for $f : \mathbb{R}^+ \to \mathbb{R}^+$ we know that $f(x) = 1/x$ is pointwise continuous, but not uniformly continuous (see Example 1 in my online notes for Complex Analysis 1 [MATH 5510] on A Primer on Lipschitz Functions).

Definition. For $A \subset X$ ($(X, \|\cdot\|)$ a normed linear space), the *closure* of A is the intersection of all closed sets containing A, denoted $\overline{A}$.

Theorem 2.2.A. For $A \subset X$ we have

(i) $\overline{A}$ equals the union of A and its boundary points.

(ii) $\overline{A} = \{x \in X \mid \text{ for all } r > 0, B(x;r) \cap A \neq \varnothing\}$.

(iii) $\overline{A} = \{x \in X \mid \text{ there is a sequence } (a_n) \text{ in } A \text{ with } (a_n) \to x\}$.

(iv) $\overline{A} = \{x \in X \mid d(x, A) = 0\}$.

The proof is to be given in Exercise 2.5.

Note. We now justify what seems obvious (given our suggestive verbiage and notation).

Proposition 2.5. With the notation above,

(i) $B(x; r)$ is open where $r > 0$.

(ii) The closure of $B(x; r)$ is $\overline{B}(x; r)$.

Definition. In a normed linear space X, set $Y \subseteq X$ is *dense* in X if $\overline{Y} = X$.

Note. In the normed linear space $X = \mathbb{R}^n$, the set $Y = \mathbb{Q}^n$ is dense in X.

Note. By Theorem 2.2.A(iii), if Y is dense in X, then for all $x \in X$ there exists a sequence $(y_n) \subseteq Y$ such that $(y_n) \to x$. This means that if f is continuous on X, then the values of f on X can be determined by the values of f on Y, since $f(x) = \lim f(y_n)$. This will be useful later, especially when Y is a *countable* dense subset of X (see Subsection 2.9.7, "ℓ^p Spaces").

Note. The text comments (page 17): "One of the most important concepts of analysis is that of *compactness*." You are familiar with this in the setting of $\mathbb{R}$ from your Analysis 1 (MATH 4217/5217) experience (see my online notes for this class on Section 3.1. Topology of the Real Numbers). However, there are certain properties of compact sets in $\mathbb{R}$ (namely, the Heine-Borel Theorem) which do not necessarily hold in more general settings, such as normed linear spaces.

Definition. A set K, a subset of a normed linear space, is *compact* if either of the following two equivalent properties hold:

(i) Given any collection of open sets with union containing K, there is a finite subcollection of these sets with union containing K. The collection of open sets is called an *open cover* of K.

(ii) For any sequence $(x_n) \subseteq K$, there is a subsequence (x_{n_k}) which converges to a point in K. Sometimes (Royden and Fitzpatrick, Section 9.5. Compact Metric Spaces) this is called *sequentially compact*.

Note. The equivalence of (i) and (ii) above is established in the metric space setting in Theorem 9.16 of Royden and Fitzpatrick's *Real Analysis* (4th Edition, Boston: Prentice Hall, 2010).

Theorem 2.2.B. The Compact Set Theorem.

If $K \subseteq X$, X a normed linear space, is compact then K is closed and bounded.

Note. The Heine-Borel Theorem states that a set *of real numbers* is compact if and only if it is closed and bounded. In fact, the result holds as well in $\mathbb{R}^n$. However, there are settings where Heine-Borel does not hold. By Theorem 2.2.B, we know that compact sets are always closed and bounded. In the infinite dimensional Hilbert space ℓ^2, there is a closed and bounded set that is not compact. To paraphrase, we can say that the Heine-Borel Theorem holds in *finite dimensional spaces*.

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