2.3. Space of Bounded Functions

Note. We now introduce our first infinite dimensional normed linear space. We let S be an arbitrary set. Recall that $F(S)$ is the set of all $f : S \to \mathbb{R}$.

Definition. For $f \in F(S)$, define

$$\|f\| = \sup\{|f(s)| \mid s \in S\}.$$

$\|\cdot\|$ is called the *sup norm*. Define the set of all bounded functions on S as

$$B(S) = \{f \in F(S) \mid \|f\| < \infty\}.$$

Theorem 2.3.A. $B(S)$ is a normed linear space.

Note. In Exercise 1.8, it is shown that $B(S)$ under $\|\cdot\|$ is infinite dimensional when S is infinite.

Note 2.3.A. If $(f_n) \to f$ in this normed linear space, then for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\|f_n - f\| < \varepsilon$. Notice that $\|f_n - f\| = \sup\{|f_n(s) - f(s)| \mid s \in S\}$. So for all $n \geq N$, we have $|f_n(s) - f(s)| < \varepsilon$ for all $s \in S$. This is uniform convergence of (f_n) to f. Therefore, convergence under the sup norm implies uniform convergence.

Revised: 5/19/2021