2.4. Bounded Linear Operators

Note. In this section, we consider operators. Operators are mappings from one normed linear space to another. We define a norm for an operator. In Chapter 6 we will form a linear space out of the operators (called a *dual space*).

Definition. For normed linear spaces X and Y, the set of all linear operators from X to Y is denoted $\mathcal{L}(X,Y)$. For $T \in \mathcal{L}(X,Y)$ define the *operator norm*

$$\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| = 1\}.$$

If $\|T\| < \infty$, then T is *bounded.* Denote the set of all bounded operators in $\mathcal{L}(X,Y)$ as $\mathcal{B}(X,Y)$.

Note 2.4.A. For any nonzero $x \in X$, we have that $x/\|x\|$ is a unit vector. So by the definition of $\|T\|$ we have

$$\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| = 1\} \geq \|T(x/\|x\|)\| = \|T(x)\|/\|x\|.$$

So for all $x \in X$, $\|Tx\| \leq \|T\|\|x\|$. We will often use this inequality when discussing operator norms.

Note. Since we use norms to measure lengths, we can use the operator norm to see if the operator preserves lengths.

Definition. A linear operator $T \in \mathcal{L}(X,Y)$ is an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$.

Note. Since X and Y are normed, for $T \in \mathcal{L}(X,Y)$ we can define continuity for T in terms of epsilons and deltas.

Definition. For $T \in \mathcal{L}(X,Y)$, T is *continuous* at $x_0 \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x_0 - x\| < \delta$ then $\|T(x_0) - T(x)\| < \varepsilon$ (that is, $T(B(x_0;\delta)) \subseteq B(T(x_0);\varepsilon)$). T is *uniformly continuous* on X (or a subset of X) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_1 - x_2\| < \delta$ implies $\|T(x_1) - T(x_2)\| < \varepsilon$ for all $x_1, x_2 \in X$ (or in the subset of X).

Theorem 2.6. Given $T \in \mathcal{L}(X,Y)$, the following are equivalent:

(i) T is uniformly continuous on X;

(ii) T is continuous at some point $x \in X$;

(iii) T is bounded.

Example 2.7. Consider the set of bounded functions on set S, $B(S)$, with the sup norm. For $g \in B(S)$ define the linear operator $M_g : B(S) \to B(S)$ as $M_g(f(s)) = g(s)f(s)$. M_g is a *multiplication operator*. We claim that $\|M_g\| = \|g\|$.

Let $f \in B(S)$ with $\|f\| = 1$. Then

$$|M_g(f)(s)| = |g(s)f(s)| = |g(s)||f(s)| \text{ (for all } s \in S) \leq \|g\|\|f\| = \|g\|.$$

Take a supremum over all $s \in S$ to get $\|M_g(f)\| \le \|g\|$. Take a supremum over all such $\|f\| = 1$, to get $\|M_g\| \le \|g\|$. Also, for all $s \in S$, we consider the characteristic function

$$\chi_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \neq s \end{cases}$$

(the text denotes this as δ_s) and we have by Note 2.4.A

$$|g(s)| = |(M_g \chi_s)(s)| \le \|M_g\| \|\chi_s\| = \|M_g\|(1) = \|M_g\|.$$

Therefore, $\sup_{s \in S} |g(s)| = \|g\| \le \|M_g\|$, and $\|M_g\| = \|g\|$ follows.

Definition. If $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ are bounded linear operators, then the *composition* of S and T is defined as $ST : X \to Z$ where $(S \circ T)(x) = ST(x) = S(T(x))$.

Note. Compositions of continuous functions are continuous, so for S and T continuous, $S \circ T$ is continuous. So, by Theorem 2.6, for bounded S and T, we have that $S \circ T = ST$ is bounded. The following result gives a specific bound on $\|ST\|$ in terms of $\|S\|$ and $\|T\|$.

Proposition 2.8. For $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ bounded linear operators, $S \circ T = ST$ is linear and $\|ST\| \le \|S\| \|T\|$.

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