2.4. Bounded Linear Operators

Note. In this section, we consider operators. Operators are mappings from one normed linear space to another. We define a norm for an operator. In Chapter 6 we will form a linear space out of the operators (called a dual space).

Definition. For normed linear spaces $X$ and $Y$, the set of all linear operators from $X$ to $Y$ is denoted $\mathcal{L}(X,Y)$. For $T \in \mathcal{L}(X,Y)$ define the operator norm

$$\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| = 1\}.$$

If $\|T\| < \infty$, then $T$ is bounded. Denote the set of all bounded operators in $\mathcal{L}(X,Y)$ as $\mathcal{B}(X,Y)$.

Note. For any nonzero $x \in X$, we have that $x/\|x\|$ is a unit vector. So by the definition of $\|T\|$ we have

$$\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| = 1\} \geq \|T(x/\|x\|)\| = \|T(x)\|/\|x\|.$$

So for all $x \in X$, $\|Tx\| \leq \|T\|\|x\|$. We will often use this inequality when discussing operator norms.

Note. Since we use norms to measure lengths, we can use the operator norm to see if the operator preserves lengths.
Definition. A linear operator \( T \in \mathcal{L}(X, Y) \) is an isometry if \( \|Tx\| = \|x\| \) for all \( x \in X \). That is, if \( \|T\| = 1 \).

Note. Since \( X \) and \( Y \) are normed, for \( T \in \mathcal{L}(X, Y) \) we can define continuity for \( T \) in terms of epsilons and deltas.

Definition. For \( T \in \mathcal{L}(X, Y) \), \( T \) is continuous at \( x_0 \in X \) if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( \|x_0 - x\| < \delta \) then \( \|T(x_0) - T(x)\| < \epsilon \) (that is, \( T(B(x_0; \delta) \subseteq B(T(x_0); \epsilon)) \)). \( T \) is uniformly continuous on \( X \) (or a subset of \( X \)) if for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \|x_1 - x_2\| < \delta \) implies \( \|T(x_1) - T(x_2)\| < \epsilon \) for all \( x_1, x_2 \in X \) (or in the subset of \( X \)).

Theorem 2.6. Given \( T \in \mathcal{L}(X, Y) \), the following are equivalent:

(i) \( T \) is uniformly continuous on \( X \);

(ii) \( T \) is continuous at some point \( x \in X \);

(iii) \( T \) is bounded.

Example 2.7. Consider the set of bounded functions on set \( S \), \( B(S) \), with the sup norm. For \( g \in B(S) \) define the linear operator \( M_g : B(S) \to B(S) \) as \( M_g(f(s)) = g(s)f(s) \). \( M_g \) is a multiplication operator. We claim that \( \|M_g\| = \|g\| \).

Let \( f \in B(S) \) with \( \|f\| = 1 \). Then

\[
|M_g(f)(s)| = |g(s)f(s)| = |g(s)||f(s)| \quad (\text{for all } s \in S) \leq \|g\||f\| = \|g\|.
\]
2.4. Bounded Linear Operators

Take a supremum over all \( s \in S \) to get \( \|M_g f\| \leq \|g\| \). Take a supremum over all such \( \|f\| = 1 \), to get \( \|M_g\| \leq \|g\| \). Also, for all \( s \in S \), we consider the characteristic function

\[
\chi_s(x) = \begin{cases} 
1 & \text{if } x = s \\
0 & \text{if } x \neq s
\end{cases}
\]

(the text denotes this as \( \delta_s \)) and we have

\[
|g(s)| = |(M_g \chi_s)(s)| \leq \|M_g\| \|\chi_s\| = \|M_g\|(1) = \|M_g\|.
\]

Therefore, \( \sup_{s \in S} |g(s)| = \|g\| \leq \|M_g\| \), and \( \|M_g\| = \|g\| \) follows.

**Definition.** If \( T \in \mathcal{L}(X, Y) \) and \( S \in \mathcal{L}(Y, Z) \) are bounded linear operators, then the composition of \( S \) and \( T \) is defined as \( ST : X \to Z \) where \( (S \circ T)(x) = ST(x) = S(T(x)) \).

**Note.** Compositions of continuous functions are continuous, so for \( S \) and \( T \) continuous, \( S \circ T \) is continuous. So, by Theorem 2.6, for bounded \( S \) and \( T \), we have that \( S \circ T \) is bounded. The following result gives a specific bound on \( \|ST\| \) in terms of \( \|S\| \) and \( \|T\| \).

**Proposition 2.8.** For \( T \in \mathcal{L}(X, Y) \) and \( S \in \mathcal{L}(Y, Z) \) bounded linear operators, \( S \circ T \) is linear and \( \|ST\| \leq \|S\| \|T\| \).

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