2.5. Completeness

Note. Recall that the Axiom of Completeness for the real numbers (an ordered field) is: Every set of real numbers with an upper bound has a least upper bound. See my online notes on Analysis 1 (MATH 4217/5217) on Section 1.3. The Completeness Axiom. Of course, this explicitly uses the ordering of $\mathbb{R}$ (in that it refers to "upper" and "least"). The idea of completeness in any space is that there are "no holes" in the space. However, a formal mathematical definition is required. In the absence of an ordering (such as in $\mathbb{C}$ for which there is no ordering), however, we need another approach to completeness. Recall, in $\mathbb{R}$, that a sequence is Cauchy if and only if it is convergent (see Exercises 2.3.13 and 2.3.14 in my online notes for Analysis 1 on Section 2.3. Bolzano-Weierstrass Theorem). In $\mathbb{R}$, this is equivalent to the Axiom of Completeness (that is, we can assume that Cauchy sequences converge and then show that sets with upper bounds have least upper bounds…by the way, the Triangle Inequality implies that convergent sequences are Cauchy, as we'll show below).

Definition. A sequence (x_n) in a normed linear space is *Cauchy* if given $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $\|x_m - x_n\| < \varepsilon$.

Definition. A normed linear space is *complete* if every Cauchy sequence converges. A complete normed linear space is called a *Banach space*.

Proposition 2.9. In a normed linear space:

(a) A convergent sequence is Cauchy.

(b) A Cauchy sequence (x_n) is bounded. That is, there is a $k > 0$ such that $\|x_n\| < k$ for all n.

(c) All subsequences of a Cauchy sequence are Cauchy.

(d) If (x_n) is Cauchy and some subsequence converges to x, then (x_n) converges to x.

Note. The above result still does not allow us to test a sequence for Cauchy-ness, but instead gives us properties of Cauchy sequences. The following definition will, as shown in Proposition 2.10, give us a technique for testing a sequence for Cauchy-ness, beyond the definition.

Definition. A sequence (x_n) is a *fast Cauchy sequence* if $\|x_{n+1} - x_n\| \leq 1/2^n$ for $n \in \mathbb{N}$.

Note. The sequence $\{1/2^n\}$ used in the definition of "fast Cauchy sequence" could be replaced with any positive term sequence $\{a_n\}$ convergent to 0. In Royden's *Real Analysis* book, a sequence (x_n) is said to be *rapidly Cauchy* if there is a series of positive numbers $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ for which $\|x_{n+1} - x_n\| \leq \varepsilon_n^2$ for all $n \in \mathbb{N}$. See my online notes for Real Anlaysis 1 (MATH 5210) on Section 7.3. L^p is Complete: The Riesz-Fischer Theorem for details. However, we follow the book's definition.

Proposition 2.10. In a normed linear space:

(a) A fast Cauchy sequence is Cauchy.

(b) Any Cauchy sequence contains a fast Cauchy subsequence.

Note. By Proposition 2.9(d) and Proposition 2.10(b), it follows that a normed linear space is complete if all fast Cauchy sequences converge.

Definition. In normed linear space X, the *series* $\sum_{i=1}^{\infty} x_i$ is said to *converge* to x, denoted $\sum_{i=1}^{\infty} x_i = x$, if the sequence of *partial sums* (s_n), where $s_n = \sum_{i=1}^{n} x_i$, converges to x.

Proposition 2.11. If $x = \sum_{i=1}^{\infty} x_i$ exists, then $\|x\| \leq \sum_{i=1}^{\infty} \|x_i\|$.

Definition. The series $\sum_{i=1}^{\infty} x_i$ is *absolutely convergent* if the series $\sum_{i=1}^{\infty} \|x_i\|$ converges.

Note. The following result gives another way to establish completeness.

Theorem 2.12. A normed linear space X is complete if and only if every absolutely convergent series is convergent.

Note. The following result will be useful in establishing the convergence of a Cauchy sequence.

Lemma 2.13. Suppose that X is a subspace of the space of all functions from set S to field $\mathbb{F}$, $F(S)$, and that $\|\cdot\|$ is a norm on X for which the closed unit ball $\overline{B}(1)$ is closed under pointwise limits. That is, if Cauchy sequence $(f_n) \subset \overline{B}(1)$ converges pointwise to f, then $f \in X$ and $f \in \overline{B}(1)$. If a sequence (f_n) in X is Cauchy and converges pointwise to f, then $f \in X$ and (f_n) converges to f with respect to $\|\cdot\|$.

Note. The following two "theorems" are actually just examples of complete normed linear spaces.

Theorem 2.14. The space of all bounded functions from set S to field $\mathbb{F}$ (taken to be $\mathbb{R}$ or $\mathbb{C}$), $B(S)$, is complete with respect to the sup norm.

Theorem 2.15. Let X and Y be normed linear spaces and suppose that Y is complete. Then the space of all bounded linear operators from X to Y, $\mathcal{B}(X, Y)$, is complete.

Idea of Proof. Similar to the proof of Theorem 2.14 where pointwise convergence and the completeness of Y is used to find a candidate limit of a Cauchy sequence.

Theorem 2.16. A subspace Y of a Banach space X is itself a Banach space if and only if Y is closed.

Example 2.17. Consider $[a,b]$. By Theorem 2.14 $B[a,b]$ (the linear space of all bounded functions on $[a,b]$) is a Banach space. Consider $C[a,b]$, the linear space of all continuous functions on $[a,b]$. A sequence $(f_n) \subset C[a,b]$ that is convergent with respect to the sup norm is, in fact, uniformly convergent by Note 2.3.A. So the limit of (f_n) is itself a continuous function (see Theorem 8.2 in my online notes for Analysis 1 [MATH 4217/5217] on Section 8.1. Sequences of Functions) and so $C[a,b]$ is closed. So by Theorem 2.16, $C[a,b]$ is a Banach space under the sup norm.

Note. The book calls the following an "extension theorem." It involves extending a bounded linear operator from a dense set X_0 in X to all of X.

Theorem 2.20. Extension Theorem.
Suppose that X_0 is a dense subspace of the normed linear space X such that $T_0 \in \mathcal{B}(X_0, Z)$ (i.e., T_0 is a bounded linear operator from X_0 to Z), where Z is a Banach space. Then T_0 has a unique extension to an operator $T \in \mathcal{B}(X, Z)$. Moreover, $\|T\| = \|T_0\|$, and if T_0 is an isometry, then so is T.

Note. Similar to the extension of a bounded linear operator from dense subset to the whole space, we can start with a normed linear space and "complete" it in the sense of having the given space as a dense subspace of some complete space.

Definition. Given any normed linear space X, a *completion* of X consists of a Banach space $\tilde{X}$ and an isometry $J : X \to \tilde{X}$ such that $J(X)$ is dense in $\tilde{X}$.

Example. $\mathbb{R}$ is a completion of $\mathbb{Q}$ where $J(x) = x$ for all $x \in \mathbb{Q}$.

Theorem 2.22. Completion Theorem.

For any normed linear space, a completion exists. Moreover, the completion is unique in the following sense: If $(\tilde{X}_1, J_1)$ and $(\tilde{X}_2, J_2)$ are completions of X, there is a surjective (onto) isometry $U : \tilde{X}_1 \to \tilde{X}_2$ such that $UJ_1 = J_2$.

Note. In fact, Theorem 2.22 holds in the more general setting of metric spaces. See my online notes for Introduction to Topology (MATH 4357/5357) on Section 7.43. Complete Metric Spaces; see Theorem 43.7. We rely on this result to establish the existence part of Theorem 2.22.

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