2.6. Comparisons of Norms

Note. In this section, we consider a linear space X with two norms on it. We compare the norms in terms of convergent sequences.

Definition. Given two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a linear space X, we say that $\|\cdot\|_1$ is *weaker* than $\|\cdot\|_2$ if any sequence convergent with respect to $\|\cdot\|_2$ is also convergent with respect to $\|\cdot\|_1$. In this case, we say that $\|\cdot\|_2$ is *stronger* than $\|\cdot\|_1$.

Note. Think of "strength" as the ability to stop sequences from converging! If $\|\cdot\|_1$ can stop a sequence from converging and $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$, then certainly $\|\cdot\|_2$ can also stop the sequence from converging. But there may be sequences for which $\|\cdot\|_2$ is strong enough to stop convergence but for which $\|\cdot\|_1$ is not strong enough to stop convergence. The next proposition makes this more quantitative and easier to deal with than this non-rigorous comment!

Proposition 2.23. $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ if and only if there is $K > 0$ such that $\|x\|_1 \le K\|x\|_2$ for all $x \in X$.

Note. The idea of stronger and weaker norms is related to the idea of stronger and weaker topologies in the setting of topological spaces (also called "finer" and "coarser" topologies). See my online notes for Introduction to Topology (MATH 4357/5357) on Section 12. Topological Spaces. The following proposition and corollary deal directly with open balls and open sets under different norms.

Proposition 2.24. $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ if and only if every $\|\cdot\|_1$ open ball contains a $\|\cdot\|_2$ open ball.

Corollary 2.25. $\|\cdot\|_1$ is weaker than $\|\cdot\|_2$ if and only if every $\|\cdot\|_1$ open set is $\|\cdot\|_2$ open.

Note. The text says that "...the space with the weaker norm has more convergent sequences, has bigger closures, has fewer continuous functions from the space, and admits more continuous functions to the space."

Definition. Two norms are *equivalent* if each is weaker than the other.

Note. By Proposition 2.24 and Corollary 2.25, two equivalent norms on a space have the same open and closed sets, the same convergent sequences, and admit the same continuous functions to and from the space. The following addresses completeness.

Theorem 2.26. If X is a Banach space with respect to a norm $\|\cdot\|_1$, it is also a Banach space with respect to any equivalent norm.

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