2.7. Quotient Spaces

Note. In modern algebra, you study the quotient group of a group G with respect to a normal subgroup N, denoted G/N, read "G mod N." The elements of G/N are the cosets of N: $G/N = \{g + N \mid g \in G\}$. As shown in Introduction to Modern Algebra (MATH 4127/5127), the cosets are equivalence classes of an equivalence relation. See my online class notes for Introduction to Modern Algebra (MATH 4127/5127) on Section II.10. Cosets and the Theorem of Lagrange and Section III.14. Factor Groups, and my online notes for Modern Algebra (MATH 5410) on Section I.5. Normality, Quotient Groups, and Homomorphisms.

Definition. Let N be a subspace of a linear space X. For $x, y \in X$, define $x \sim y$ if $(x - y) \in N$. Denote the equivalence class under $\sim$ which contains x as $\overline{x}$.

Note. $\sim$ is an equivalence relation on X. Therefore, the equivalence classes of $\sim$ partition X. For $x \in X$, denote the equivalence class containing x as $\overline{x}$. The equivalence classes are called *linear manifolds*. With $X = \mathbb{R}^2$ and N a one dimensional subspace of X, we see that the equivalence classes on X form the collection of lines parallel to N in $\mathbb{R}^2$ (when all vectors are put in standard position):

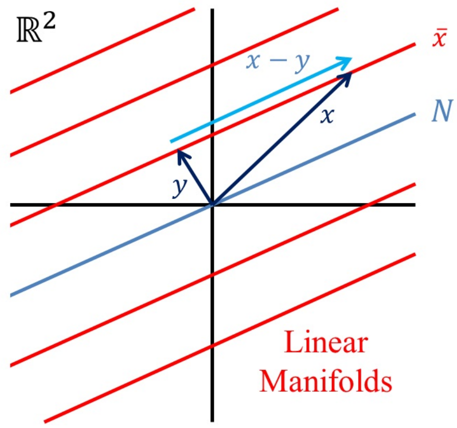

Figure 2.7

Definition. The set of all equivalence classes is denoted X/N (read "X mod N"). Define *addition* and *scalar multiplication* on X/N as $\overline{x} + \overline{y} = \overline{x + y}$ and $\alpha\overline{x} = \overline{\alpha x}$.

Note. That is, addition and scalar multiplication are defined in terms of representatives of equivalence classes. So it should be confirmed that these two operations are well defined. This is to be done in Exercise 2.A. WARNING: Do not confuse the overline with complex conjugation!

Definition. The linear space X/N is a *quotient space* of X.

Definition. Define $\pi_N : X \to X/N$ as $\pi_N(x) = \overline{x}$. π_N is the *canonical projection* with respect to N.

Note. In the setting of quotient groups, the mapping π_N is called the "canonical homomorphism" (see my online Introduction to Modern Algebra notes on Section III.14. Factor Groups; see Theorem 14.9 and Theorem 14.11 "The Fundamental Homomorphism Theorem") or the "canonical epimorphism" (see my online Modern Algebra [MATH 5410] notes on Section I.5. Normality, Quotient Groups, and Homomorphisms).

Lemma 2.7.A. The canonical projection π_N is an onto (surjective) linear operator.

Note. A proof of Lemma 2.7.A is to be given in Exercise 2.7.B.

Note 2.7.A. Given linear $T : X \to Y$ with nullspace $N(T)$, we define a mapping $\tilde{T} : X/N(T) \to Y$ as $\tilde{T}\overline{x} = Tx$. Since T is linear, then $\tilde{T}$ is linear. Notice $\tilde{T}$ is one to one (injective) since, for $\tilde{T}\overline{x} = Tx = \tilde{T}\overline{y} = Ty$, we have that $Tx - Ty = T(x-y) = 0$, so $x - y \in N(T)$, and $x \sim y$ or $\overline{x} = \overline{y}$. Similarly, $x \sim y$ implies $\overline{x} = \overline{y}$ and so $\tilde{T}\overline{x} = \tilde{T}\overline{y}$. Therefore, $\tilde{T}\overline{x} = \tilde{T}\overline{y}$ if and only if $x \sim y$. Since $\pi_{N(T)} : X \to X/N(T)$ is onto (surjective) by Lemma 2.7.A, then when we write $T = \tilde{T}\pi_{N(T)}$, we have expressed T as the composition of a one to one mapping (namely, mapping $\tilde{T}$) with an onto mapping (namely, mapping $\pi_{N(T)}$).

Definition. For $\overline{x} \in X/N$, define the norm on X/N as

$$\|\overline{x}\| = \inf\{\|x - z\| \mid z \in N\} = d(x, N).$$

Note. If $x, y \in \overline{x}$ then $x - y \in N$ and so

$$\inf\{\|x - z\| \mid z \in N\} = \inf\{\|x - (z + x - y) \mid z \in N\} \text{ since } N \text{ is a subspace}$$

$$= \inf\{\|y - z\| \mid z \in N\},$$

and so $\|\overline{x}\|$ is well-defined.

Theorem 2.27. Let N be a closed subspace of the normed linear space X.

(a) The quantity $\|\overline{x}\|$ defines a norm on X/N.

(b) If X is a Banach space, then X/N is a Banach space.

(c) $\|\pi_N\| = 1$.

(d) If $N = N(T)$ (the nullspace of bounded linear $T : X \to Y$) then the map $\tilde{T} : X/N \to Y$ defined as $\tilde{T}\overline{x} = Tx$ has the same norm as T: $\|\tilde{T}\| = \|T\|$.

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