2.8. Finite Dimensional Normed Linear Spaces

Note. In this section, we classify a normed linear space as finite dimensional by considering the topological property of compactness.

Note. Each of the following is a norm on $\mathbb{R}^2$ (here we denote $x \in \mathbb{R}^2$ as (x_1, x_2)):

(i) the sup norm: $\|x\|_\infty = \|x\|_{\text{sup}} = \max\{|x_1|, |x_2|\}$

(ii) the Euclidean norm: $\|x\| = \sqrt{(x_1)^2 + (x_2)^2}$

(iii) the absolute value deviation norm (also called the "taxicab norm"): $\|x\|_1 = |x_1| + |x_2|$.

Notice the appearance of unit balls under these norms in Figure 2.3. In fact, we will show that all norms on a finite dimensional space are equivalent (in Theorem 2.31(b)).

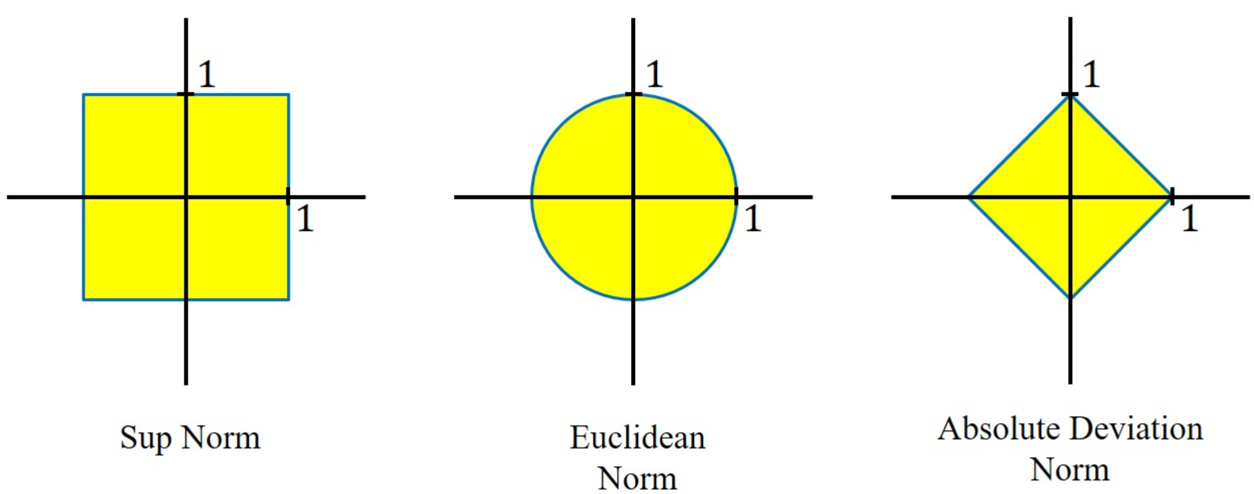

Figure 2.3. Unit balls under various norms.

Note. In this section, $F = \{1, 2, \ldots, N\}$ and $B(F)$ is the set of bounded functions on F:

$$f \in B(F) \Rightarrow f : F \to \mathbb{F} \text{ and } \{f(s) \mid s \in F\} \text{ is bounded.}$$

We equip $B(F)$ with the sup norm. From Section 1.4, $\{\delta_i \mid i = 1, 2, \ldots, N\}$ where

$$\delta_i(s) = \begin{cases} 0 & \text{if} \quad i \neq s \\ 1 & \text{if} \quad i = s \end{cases}$$

is a basis of $B(F)$ since any $f \in B(F)$ can be uniquely written as $f(x) = \sum_{i=1}^{N} f(i)\delta_i(x)$.

Note. In the next result, we show that a linear operator on a finite dimensional normed linear space is necessarily bounded. Since in sophomore Linear Algebra (MATH 2010) you almost exclusively consider finite dimensional spaces, the concept of boundedness does not arise there since all linear operators are necessarily bounded in that setting.

Proposition 2.28. For any normed linear space Z, all elements of $\mathcal{L}(B(F), Z)$ (the set of linear operators from $B(F)$ to Z) are bounded.

Note. The familiar Heine-Borel Theorem states that a *set of real numbers* is compact if and only if it is closed and bounded. We will see this violated in infinite dimensional spaces. However, "half" of Heine-Borel holds in $B(F)$. The other half holds by Theorem 2.2.B. Notice that F is a finite set here.

Theorem 2.29. Closed and bounded subsets of $B(F)$ are compact.

Proposition 2.30. If T is a bijective linear operator from the normed linear space X to $B(F)$, then T is bounded.

Note. We now prove some properties of finite dimensional spaces.

Theorem 2.31.

(a) Any linear operator $T : X \to Z$, where X is finite dimensional, is bounded.

(b) All norms on a finite dimensional space are equivalent and all finite dimensional normed linear spaces over field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$) are complete.

(c) Any finite-dimensional subspace of a normed linear space is closed.

Theorem 2.32. A linear operator $T : X \to Y$, where Y is finite-dimensional, is bounded if and only if $N(T)$ (the nullspace of T) is closed.

Note. We want to talk about "orthogonality" (or "perpendiculars"). However, in a normed linear space, we do not necessarily have an inner product ("dot product") and so we cannot measure angles. However, inspired by our experience in Euclidean space $\mathbb{R}^n$ (and the usual Euclidean norm), we have the following definition concerning perpendiculars to subspaces.

Definition. Given a proper subspace M of the normed linear space $(X, \|\cdot\|)$, a nonzero vector $x \in X$ is *perpendicular* to M if

$$d(x, M) = \inf\{\|x - y\| \mid y \in M\} = \|x\|.$$

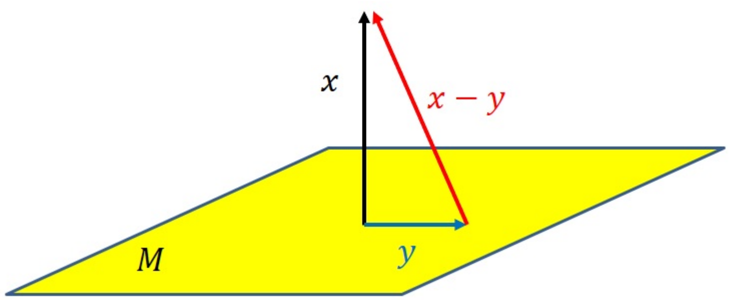

Figure. The idea of "perpendicular" defined using norms.

Note. Sometimes there are no perpendiculars to a proper subspace M of normed linear space X, as shown in Exercise 2.25. However, there are always "nearly perpendicular" vectors to M, as shown in the following.

Theorem 2.33. Riesz's Lemma.
Given a closed, proper subspace M of a normed linear space $(X, \|\cdot\|)$ and given $\varepsilon > 0$, there is a unit vector $x \in X$ such that $d(x, M) \geq 1 - \varepsilon$.

Note. The following result is particularly intriguing! It relates the algebraic idea of dimension to the topological idea of compactness. It also hints at the fact that the Heine-Borel Theorem only holds in finite dimensional spaces.

Theorem 2.34. Riesz's Theorem. A normed linear space $(X, \|\cdot\|)$ is finite-dimensional if and only if the closed unit ball $\overline{B}(0; 1)$ is compact.

Note. Riesz's Theorem is also presented in the Real Analysis sequence (MATH 5210, MATH 5220). See my online notes for those classes on Section 13.3. Infinite Dimensional Normed Linear Spaces.

Revised: 12/13/2022