2.9. \( L^p \) Spaces

**Note.** In this section, we introduce the spaces \( L^p \) and \( \ell^p \) for \( 1 \leq p \leq \infty \). We’ll see that these spaces are Banach space. To completely appreciate the development of these ideas, we need an understanding of Lebesgue measure, Lebesgue integration, and convergence properties of Lebesgue integrals. This material is covered in Real Analysis 1 (MATH 5210), but since that class is not a prerequisite for this class, we list (without proof) some of the major results concerning Lebesgue measure and integration. For a really nice development of Lebesgue’s theory, see Royden and Fitzpatrick’s *Real Analysis*, 4th Edition.

**Definition.** Given a set \( \Omega \), a collection \( S \) of subsets of \( \Omega \) is a \( \sigma \)-algebra if:

(a) Given any finite or countable infinite sequence \( A_1, A_2, \ldots \) of sets in \( S \), we have

(i) their union is in \( S \)

(ii) their intersection is in \( S \).

(b) For any \( A \in S \), the complement \( A^c \in S \).

(c) \( \Omega \in S \).

**Note.** Be DeMorgan’s Law, property (a)(i) and (b) combine to give property (a)(ii) (and similarly, (a)(ii) and (b) combine to give (a)(i)).

**Example.** The power set \( \mathcal{P}(S) \) is a \( \sigma \)-algebra on \( S \).
Example. A more interesting example of a $\sigma$-algebra on $\mathbb{R}$ is the “smallest” $\sigma$-algebra containing all open sets of real numbers, $\mathcal{B}$, which is called the set of Borel sets. However, there are not many Borel sets and in terms of cardinality $|\mathbb{R}| = |\mathcal{B}| < |\mathcal{P}(\mathbb{R})|.$

Definition. A measure $\mu$ on a $\sigma$-algebra $S$ is a function from $S$ to $[0, \infty]$ which is countable additive (that is, for $A_1, A_2, \ldots$ disjoint sets in $S$ we have $\mu(\bigcup_{k=1}^{\infty} A_i) = \sum_{k=1}^{\infty} \mu(A_k)$ and satisfies $\mu(\emptyset) = 0$. A measure space $(\Omega, S, \mu)$ where $S$ is a $\sigma$-algebra of subsets of $\Omega$ and $\mu$ is a measure on $S$. A function $f : \Omega \to \mathbb{R}$ is a measurable function if for all $r \in \mathbb{R}$, $f^{-1}((-\infty, r]) \in S$.

Note. We omit a huge number of details about Lebesgue measure. Suffice it to say that Lebesgue measure $m$ is defined on a $\sigma$-algebra of sets of real numbers, denoted $\mathcal{M}$ (the $\sigma$-algebra of Lebesgue measurable sets), for which every interval is in $\mathcal{M}$ (and so $\mathcal{M}$ contains all open and closed subsets of $\mathbb{R}$) and the Lebesgue measure of an interval is its length. In terms of cardinality, $|\mathcal{M}| = |\mathcal{P}(\mathbb{R})|$, but $\mathcal{M} \neq \mathcal{P}(\mathbb{R})$ and the Axiom of Choice can be used to “construct” a non-Lebesgue measurable set. The construction is related to the offensive Banach-Tarski paradox. See the class notes for Real Analysis 1 (MATH 5210) for more details.

Definition. A measure space $(\Omega, S, \mu)$ is finite if $\mu(\Omega) < \infty$. 
**Definition.** A *simple function* on a measure space \((\Omega, S, \mu)\) is a measurable function which takes on only a finite numbers of values. If \(s\) is simple and takes on the values \(c_1, c_2, \ldots, c_n\) then the *integral* of \(s\) over \(\Omega\) is

\[
\int_{\Omega} s \, d\mu = \sum_{k=1}^{n} c_k \mu[s^{-1}(c_k)].
\]

**Example.** Consider the (modified) Dirichlet function on \([0, 1]\)

\[
D(x) = \begin{cases} 
1 & \text{if } x \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}) \\
0 & \text{if } x \in [0, 1] \cap \mathbb{Q}.
\end{cases}
\]

Then \(\int_{[0,1]} D \, dm = 1\) (one can show that a countable set has Lebesgue measure 0). This example also illustrates the use of measure theory in probability. It allows us to claim that the probability that a number chosen at random under a uniform probability distribution between 0 and 1 is rational is 0.

**Definition.** Let \(f\) be a nonnegative real-valued measurable function. Define the *Lebesgue integral*

\[
\int_{\Omega} f \, dm = \sup \left\{ \int_{\Omega} s \, dm \mid s \text{ is simple, } s \leq f \right\}.
\]

**Definition.** A set \(B \in S\) such that \(\mu(B) = 0\) is a *null set*. A property which holds on all of \(\Omega\) except on a null set is said to hold *almost everywhere*, denoted a.e.
Theorem. If \( f \) and \( g \) are nonnegative measurable functions on \( \Omega \) then:

(1) \( \int_{\Omega} (f + g) = \int_{\Omega} f + \int_{\Omega} g \) (Additivity),

(2) For any \( \alpha \geq 0 \), \( \int_{\Omega} \alpha f = \alpha \int_{\Omega} f \) (Scalar Property),

(3) If \( f \leq g \) a.e. then \( \int_{\Omega} f \leq \int_{\Omega} g \) (Monotonicity),

(4) \( \int_{\Omega} f = 0 \) if and only if \( f < \infty \) a.e.

(5) If \( \int_{\Omega} f < \infty \) then \( f < \infty \) a.e.

Theorem. Fatou’s Lemma.

If \((f_n)\) is a sequence of nonnegative measurable functions which converge pointwise a.e. to a function \( f \), then \( f \) is measurable and

\[
\int_{\Omega} f \leq \lim \inf \int_{\Omega} f_n.
\]

Theorem. Monotone Convergence Theorem.

If \((f_n)\) is a sequence of measurable functions converging pointwise a.e. to a function \( f \) and if \( f_n \leq f_{n+1} \) a.e. for all \( n \in \mathbb{N} \), then

\[
\int_{\Omega} f = \int_{\Omega} \lim f_n = \lim \int_{\Omega} f_n.
\]
Definition. For a general real-valued measurable function $f$, define the **positive** and **negative parts** as

$$f^+(t) = \max\{f(t), 0\} \text{ and } f^-(t) = -\min\{f(t), 0\}.$$  

(Notice that $f^+(t)$ and $f^-(t)$ are both nonnegative.) If $\int_{\Omega} f^+ < \infty$ and $\int_{\Omega} f^- < \infty$ then define the **integral** of $f$ as $\int_{\Omega} f = \int_{\Omega} f^+ - \int_{\Omega} f^-$. 

Definition. For complex valued measurable $f(x)$, define the **real** and **imaginary part** of $f(x)$ as

$$\text{Re}f(x) = \text{Re}(f(x)) \text{ and } \text{Im}f(x) = \text{Im}(f(x)).$$

If $\int_{\Omega} \text{Re}(f) < \infty$ and $\int_{\Omega} \text{Im}(f) < \infty$, then define the **integral** $\int_{\Omega} f = \int_{\Omega} \text{Re}f + i\int_{\Omega} \text{Im}f$. 

**Theorem.** If $f$ and $g$ are general measurable functions on $\Omega$ then:

1. $\int_{\Omega} (f + g) = \int_{\Omega} f + \int_{\Omega} g$ (Additivity),

2. For any $\alpha \in \mathbb{F}$, $\int_{\Omega} \alpha f = \alpha \int_{\Omega} f$ (Scalar Property),

3. If $f$ and $g$ are real valued and $f \leq g$ a.e. then $\int_{\Omega} f \leq \int_{\Omega} g$ (Monotonicity).
**Definition.** Let $(\Omega, S, \mu)$ be a measure space. For $p \in [1, \infty)$, define

$$\|f\|_p = \left(\int_{\Omega} |f|^p \, d\mu\right)^{1/p}.$$ 

Let

$$\mathcal{L}^p(\Omega, S, \mu) = \{f : \Omega \to \mathbb{F} \mid f \text{ is measurable, } \|f\|_p < \infty\}.$$ 

Then $\mathcal{L}^p(\Omega, S, \mu)$ is the $\mathcal{L}^p$-space on $\Omega$.

**Proposition 2.35.** $\mathcal{L}^p(\Omega, S, \mu)$ is a linear space.

**Theorem 2.37.** Hölder’s Inequality.

For all measurable functions $f$ and $g$ with $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ we have that $fg \in \mathcal{L}^1$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

**Theorem 2.38.** Minkowski’s Inequality.

For all measurable functions $f$ and $g$ with $f, g \in \mathcal{L}^p$ where $p \in [1, \infty)$, we have $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

**Definition.** For $1 \leq p < \infty$, let $N = \{f \in \mathcal{L}^p \mid \|f\|_p = 0\}$. Let $L^p(\Omega, S, \mu)$ denote the quotient space $\mathcal{L}^p/N$. this is the $L^p$ space on $\Omega$.

**Note.** A alternate way to develop the $L^p$ spaces is to partition $\mathcal{L}^p$ into equivalence classes where two functions are in the same equivalence class if they are equal a.e.
2.9. $L^p$ Spaces

Note. We have seen that $\| \cdot \|_p$ is a norm on $L^p$ and we will see that $L^p$ is complete (that is, the $L^p$ spaces are Banach spaces).

Definition. The essential supremum of $f$ on a measure space is

$$\|f\|_{\text{ess sup}} = \inf \{ r \mid |f(x)| \leq r \text{ a.e. on } \Omega \}. $$

A measurable function is essentially bounded if $\|f\|_{\text{ess sup}} < \infty$. Let $L^\infty$ denote the set of all essentially bounded functions on $\Omega$. Define $L^\infty = L^\infty/N$ where $N$ is the subspace of all functions $f$ on $\Omega$ where $\|f\|_{\text{ess sup}} = 0$.

Note. $L^\infty$ is a normed linear space under the essential supremum. So for $f \in L^\infty$ we denote $\|f\|_{\text{ess sup}}$ as $\|f\|_\infty$.

Theorem 2.41. The Riesz Fischer Theorem.

For $1 \leq p \leq \infty$, $L^p$ is a complete normed linear space with norm $\| \cdot \|_p$. That is, $L^p$ is a Banach space.

Definition. Define the set $\ell^p$ for $p \in [1, \infty)$ as

$$\ell^p = \left\{ (x_1, x_2, \ldots) \mid x_k \in \mathbb{F}, \sum_{k=1}^\infty |x_k|^p < \infty \right\}. $$

Define the $\ell^p$ norm

$$\|(x_1, x_2, \ldots)\|_p = \left( \sum_{k=1}^\infty |x_k|^p \right)^{1/p}. $$
Define the set $\ell^\infty$ as the set of all bounded sequences of elements of $\mathbb{F}$ and define the $\ell^\infty$ norm of a bounded sequence as the least upper bound of the set of absolute values of the terms of the sequence.

**Note.** The $\ell^p$ spaces are complete normed linear spaces for $p \in [1, \infty]$. That is, they are Banach spaces.

**Definition.** A normed linear space is said to be *separable* if it contains a countable dense subset.

**Proposition 2.42.** $\ell^\infty$ is not separable.

**Theorem.** $\ell^p$ is separable for $p \in [1, \infty)$.

**Idea of Proof.** The set of all sequences consisting of a finite number of rational real numbers (or rational complex numbers if $\mathbb{F} = \mathbb{C}$) with the remaining entries equal to 0 forms a countable dense set in $\ell^p$.  

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