2.9. L^p Spaces

Note. In this section, we introduce the spaces L^p and ℓ^p for $1 \le p \le \infty$. We'll see that these spaces are Banach space. To completely appreciate the development of these ideas, we need an understanding of Lebesgue measure, Lebesgue integration, and convergence properties of Lebesgue integrals. This material is covered in Real Analysis 1 (MATH 5210), but since that class is not a prerequisite for this class, we list (without proof) some of the major results concerning Lebesgue measure and integration. For a presentation of Lebesgue measure and integration, see my online notes for Real Analysis 1. Many notes from this class are referenced in this section.

Definition. Given a set Ω, a collection S of subsets of Ω is a *σ-algebra* if:

(a) Given any finite or countable infinite sequence $A_1, A_2, \ldots$ of sets in S, we have

- **(i)** their union is in S
- **(ii)** their intersection is in S.

(b) For any $A \in S$, the complement $A^c \in S$.

(c) $\Omega \in S$.

Note. By DeMorgan's Law, property (a)(i) and (b) combine to give property (a)(ii) (and similarly, (a)(ii) and (b) combine to give (a)(i)).

Example. The power set $\mathcal{P}(S)$ is a σ-algebra on S.

Example. A more interesting example of a σ-algebra on $\mathbb{R}$ is the "smallest" σ-algebra containing all open sets of real numbers, $\mathcal{B}$, which is called the set of *Borel sets*. However, there are not many Borel sets and in terms of cardinality $|\mathbb{R}| = |\mathcal{B}| < |\mathcal{P}(\mathbb{R})|$. See Section 1.4. Open Sets, Closed Sets, and Borel Sets or Real Numbers and Supplement. The Cardinality of the Set of Lebesgue Measurable Sets.

Definition. A *measure* μ on a σ-algebra S is a function from S to $[0, \infty]$ which is countable additive (that is, for $A_1, A_2, \ldots$ disjoint sets in S we have $\mu(\cup_{k=1}^{\infty} A_i) = \sum_{k=1}^{\infty} \mu(A_k)$) and satisfies $\mu(\varnothing) = 0$. The triple (Ω, S, μ) is a *measure space* if S is a σ-algebra of subsets of Ω and μ is a measure on S. A function $f : \Omega \to \mathbb{R}$ is a *measurable function* if for all $r \in \mathbb{R}$, $f^{-1}((-\infty, r]) \in S$.

Note. We omit a huge number of details about *Lebesgue measure*. Suffice it to say that Lebesgue measure m is defined on a σ-algebra of sets of real numbers, denoted $\mathcal{M}$ (the σ-algebra of *Lebesgue measurable* sets), for which every interval is in $\mathcal{M}$ (and so $\mathcal{M}$ contains all open and closed subsets of $\mathbb{R}$) and the Lebesgue measure of an interval is its length (see, in particular, Section 2.3. The σ-Algebra of Lebesgue Measurable Sets). In terms of cardinality, $|\mathcal{M}| = |\mathcal{P}(\mathbb{R})|$ but $\mathcal{M} \neq \mathcal{P}(\mathbb{R})$ (see Supplement. The Cardinality of the Set of Lebesgue Measurable Sets). The Axiom of Choice can be used to "construct" a non-Lebesgue measurable set (Section 2.6. Nonmeasurable Sets (Roydens 3rd Edition) and Section 2.6. Nonmeasurable Sets (Royden and Fitzpatrick, 4th Edition)). The construction is related to the "intuitively offensive" Banach-Tarski paradox. See my notes on Nonmeasurable sets and the Banach-Tarski Paradox.

Definition. A measure space (Ω, S, μ) is *finite* if $\mu(\Omega) < \infty$.

Definition. A *simple function* on a measure space (Ω, S, μ) is a measurable function which takes on only a finite numbers of values. If s is simple and takes on the values $c_1, c_2, \ldots, c_n$ then the *integral* of s over Ω is

$$\int_\Omega s\, d\mu = \sum_{k=1}^{n} c_k\, \mu[s^{-1}(c_k)].$$

(See Section 4.2. Lebesgue Integration of a Bounded Measurable Function over a Set of Finite Measure.)

Example. Consider the (modified) Dirichlet function on $[0,1]$

$$D(x) = \begin{cases} 1 & \text{if} \quad x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}) \\ 0 & \text{if} \quad x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Then $\int_{[0,1]} D\, dm = 1$ (one can show that a countable set has Lebesgue measure 0). This example also illustrates the use of measure theory in probability. It allows us to claim that the probability that a number chosen at random under a uniform probability distribution between 0 and 1 is rational is 0.

Definition. Let f be a nonnegative real-valued measurable function. Define the *Lebesgue integral*

$$\int_\Omega f\, dm = \sup \left\{ \int_\Omega s\, dm \,\middle|\, s \text{ is simple}, s \le f \right\}.$$

(See Section 4.2. Lebesgue Integration of a Bounded Measurable Function over a Set of Finite Measure.)

Definition. A set $B \in S$ such that $\mu(B) = 0$ is a *null set*. A property which holds on all of Ω except on a null set is said to hold *almost everywhere*, denoted a.e.

Theorem 2.9.A. If f and g are nonnegative measurable functions on Ω then:

(1) $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$ (Additivity),

(2) for any $\alpha \geq 0$, $\int_\Omega \alpha f = \alpha \int_\Omega f$ (Scalar Property),

(3) If $f \leq g$ a.e. then $\int_\Omega f \leq \int_\Omega g$ (Monotonicity),

(4) $\int_\Omega f = 0$ if and only if $f = 0$ a.e.

(5) If $\int_\Omega f < \infty$ then $f < \infty$ a.e.

Theorem 2.9.B. Fatou's Lemma.

If (f_n) is a sequence of nonnegative measurable functions which converge pointwise a.e. to a function f, then f is measurable and

$$\int_\Omega f \leq \liminf \int_\Omega f_n.$$

(See Section 4.3. Integrals of Measurable Nonnegative Function.)

Theorem 2.9.C. Monotone Convergence Theorem.

If (f_n) is a sequence of measurable functions converging pointwise a.e. to a function f and if $f_n \leq f_{n+1}$ a.e. for all $n \in \mathbb{N}$, then

$$\int_\Omega f = \int_\Omega \lim f_n = \lim \int_\Omega f_n.$$

(See Section 4.3. Integrals of Measurable Nonnegative Function.)

Definition. For a general real-valued measurable function f, define the *positive* and *negative parts* as

$$f^+(t) = \max\{f(t), 0\} \text{ and } f^-(t) = -\min\{f(t), 0\}.$$

(Notice that $f^+(t)$ and $f^-(t)$ are both nonnegative.) If $\int_\Omega f^+ < \infty$ and $\int_\Omega f^- < \infty$ then define the *integral* of f as $\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-$. (See Section 4.4. The General Lebesgue Integral.)

Definition. For complex valued measurable $f(x)$, define the *real* and *imaginary part* of $f(x)$ as

$$\mathrm{Re} f(x) = \mathrm{Re}(f(x)) \text{ and } \mathrm{Im} f(x) = \mathrm{Im}(f(x)).$$

If $\int_\Omega \mathrm{Re}(f) < \infty$ and $\int_\Omega \mathrm{Im}(f) < \infty$, then define the *integral*

$$\int_\Omega f = \int_\Omega \mathrm{Re}(f) + i \int_\Omega \mathrm{Im}(f).$$

Theorem 2.9.D. If f and g are general measurable functions on Ω then:

(1) $\int_\Omega (f + g) = \int_\Omega f + \int_\Omega g$ (Additivity),

(2) For any $\alpha \in \mathbb{F}$, $\int_\Omega \alpha f = \alpha \int_\Omega f$ (Scalar Property),

(3) If f and g are real valued and $f \leq g$ a.e. then $\int_\Omega f \leq \int_\Omega g$ (Monotonicity).

Note. The next several results are introduced in Analysis 2 (MATH 5220) in "Chapter 7. The L^p Spaces: Completeness and Approximation" (see my online notes for Part 1 of Royden and Fitzpatrick's book). They are covered in more depth in the measure space setting in "Chapter 19. General L^p Spaces: Completeness, Duality, and Weak Convergence" (see my online notes for Part 3 of Royden and Fitzpatrick's book).

Definition. Let (Ω, S, μ) be a measure space. For $p \in [1, \infty)$, define

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}.$$

Let

$$\mathcal{L}^p(\Omega, S, \mu) = \{f : \Omega \to \mathbb{F} \mid f \text{ is measurable}, \|f\|_p < \infty\}.$$

Then $\mathcal{L}^p(\Omega, S, \mu)$ is the *$\mathcal{L}^p$-space* on Ω.

Proposition 2.35. $\mathcal{L}^p(\Omega, S, \mu)$ is a linear space.

Theorem 2.37. Hölder's Inequality.

For all measurable functions f and g with $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$, where $1/p+1/q = 1$, we have that $fg \in \mathcal{L}^1$ and $\|fg\|_1 \le \|f\|_p \|g\|_q$.

Theorem 2.38. Minkowski's Inequality.

For all measurable functions f and g with $f, g \in \mathcal{L}^p$ where $p \in [1, \infty)$, we have $\|f + g\|_p \le \|f\|_p + \|g\|_p$.

Definition. For $1 \leq p < \infty$, let $N = \{f \in \mathcal{L}^p \mid \|f\|_p = 0\}$. Let $L^p(\Omega, S, \mu)$ denote the quotient space $\mathcal{L}^p/N$. This is the L^p space on Ω.

Note. A alternate way to develop the L^p spaces is to partition $\mathcal{L}^p$ into equivalence classes where two functions are in the same equivalence class if they are equal a.e.

Note. We have seen that $\|\cdot\|_p$ is a norm on L^p and we will see that L^p is complete (that is, the L^p spaces are Banach spaces).

Definition. The *essential supremum* of f on a measure space is

$$\|f\|_{\text{ess sup}} = \inf\{r \mid |f(x)| \leq r \text{ a.e. on } \Omega\}.$$

A measurable function is *essentially bounded* if $\|f\|_{\text{ess sup}} < \infty$. Let $\mathcal{L}^\infty$ denote the set of all essentially bounded functions on Ω. Define $L^\infty = \mathcal{L}^\infty/N$ where N is the subspace of all functions f on Ω where $\|f\|_{\text{ess sup}} = 0$.

Note. L^∞ is a normed linear space under the essential supremum. So for $f \in \mathcal{L}^\infty$ we denote $\|f\|_{\text{ess sup}}$ as $\|f\|_\infty$.

Theorem 2.41. The Riesz-Fischer Theorem.

For $1 \leq p \leq \infty$, L^p is a complete normed linear space with norm $\|\cdot\|_p$. That is, L^p is a Banach space.

Note. The next results concern the "little ℓ p space" and are covered in Analysis 2 (MATH 5220) in "Chapter 7. The L^p Spaces: Completeness and Approximation" (see my online notes for Part 1 of Royden and Fitzpatrick's book), mostly in homework exercises.

Definition. Define the set ℓ^p for $p \in [1, \infty)$ as

$$\ell^p = \left\{ (x_1, x_2, \ldots) \;\middle|\; x_k \in \mathbb{F}, \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}.$$

Define the ℓ^p norm

$$\|(x_1, x_2, \ldots)\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}.$$

Define the set ℓ^∞ as the set of all bounded sequences of elements of $\mathbb{F}$ and define the ℓ^∞ norm of a bounded sequence as the least upper bound of the set of absolute values of the terms of the sequence.

Note. The ℓ^p spaces are complete normed linear spaces for $p \in [1, \infty]$. That is, they are Banach spaces.

Definition. A normed linear space is said to be *separable* if it contains a countable dense subset.

Proposition 2.42. ℓ^∞ is not separable.

Theorem 2.9.E. ℓ^p is separable for $p \in [1, \infty)$.

Idea of Proof. The set of all sequences consisting of a finite number of rational real numbers (or rational complex numbers if $\mathbb{F} = \mathbb{C}$) with the remaining entries equal to 0 forms a countable dense set in ℓ^p.

Revised: 6/2/2021