3.2. Baire Category Theorem

Note. In this section, we explore some topological properties of sets, with particular interest towards intersections of dense subsets.

Note. The following is parallel to a familiar result from $\mathbb{R}$. In particular, see Theorem 3.8 in my online notes for Analysis 1 (MATH 4217/5217) on Section 3.1. Topology of the Real Numbers.

Proposition 3.1. Nested Set Theorem.
Given a sequence $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ of closed nonempty sets in a Banach space such that $\text{diam}(F_n) \to 0$, there is a unique point that is in F_n for all $n \in \mathbb{N}$.

Note. If sets A and B are dense in X, then $A \cap B$ may not be dense in X. Consider, for example, $X = \mathbb{R}$, $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$ (then $A \cap B = \varnothing$).

Note 3.2.A. Recall that set $Y \subset X$ is dense in X if $\overline{Y} = X$. Now $\overline{Y} = \{x \in X \mid \text{ for all } r > 0, B(x;r) \cap Y \neq \varnothing\}$ by Theorem 2.2.A(ii). So set Y is dense in X if and only if Y intersects every nonempty open subset of X.

Note. The following result shows that the intersections of open dense sets are dense.

Theorem 3.2. Baire's Theorem.

The intersection of countably many open and dense sets in a Banach space is dense.

Note 3.2.B. If A is a dense set in set X, then $X \setminus A$ has an empty interior. If not, then there would be some $B(x;\varepsilon) \subset X \setminus A$ but then A would not have x as a point of closure and A would not be dense in X. Conversely, if $X \setminus A$ has an empty interior, then A is dense in X.

Corollary 3.3. Dual Form of Baire's Theorem.

In any Banach space, a countable union of closed sets with empty interiors has an empty interior.

Note. The text mentions a surprising result about spaces of continuous functions. Exercise 3.14 states: "The set of nowhere differentiable functions is dense in $C[0,1]$ (the linear space of all continuous real valued functions on $[0,1]$ under the sup norm)." What makes this surprising is that you are probably familiar with very few (if any) continuous nowhere differentiable functions, yet there must be lots of them if they form a dense subset in $C[0,1]$, as claimed. You can find an example of such a function in James R. Munkres' *Topology*, 2nd Edition (Prentice Hall, 2000); see "Section 49. A Nowhere-Differentiable Function" in "Chapter 8. Baire Spaces and Dimension Theory."

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