3.3. Open Mappings

Note. This section gives a proof of the Open Mapping Theorem which states that an onto bounded linear operator maps open sets to open sets.

Lemma 3.4. Given normed linear spaces X and Y, subsets A and C of X, $\alpha \in \mathbb{F}$, and $T \in \mathcal{L}(X, Y)$, we have:

(a) $\alpha B(0; r) = \alpha B(r) = B(|\alpha| r) = B(0; |\alpha| r)$.

(b) $\alpha \overline{A} = \overline{\alpha A}$.

(c) $\overline{A} + \overline{C} \subseteq \overline{A + C}$.

(d) $T(\alpha A) = \alpha T(A)$.

Note. The proof of Lemma 3.4 is left as Exercise 3.1.

Theorem 3.5. Open Mapping Theorem.

Given a surjective (onto) $T \in \mathcal{B}(X, Y)$ where X and Y are Banach spaces, if $U \subseteq X$ is open then $T(U)$ is open.

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