3.4. Bounded Inverses

Note. In this section, we give a condition equivalent to the inverse of a bounded linear operator being itself bounded (and hence continuous).

Definition. A linear operator T from one normed linear space to another is *bounded below* if there is a $k > 0$ such that for all unit vectors x we have $\|Tx\| \geq k$.

Note 3.4.A. Notice that if T is bounded below by k and if $x \in X$ is any nonzero vector, then $\|T(x/\|x\|)\| \geq k$ and so $\|Tx\| \geq k\|x\|$.

Note. Of course, to discuss T^{-1}, T must be injective (one to one).

Theorem 3.6. Given an injective $T \in \mathcal{B}(X, Y)$ for which both X and Y are Banach spaces, the following are equivalent:

(i) T^{-1} is bounded;

(ii) T is bounded below;

(iii) $R(T)$ (the range of T) is closed.

Theorem. Bounded Inverse Theorem.

Given bijective $T \in \mathcal{B}(X, Y)$ where X and Y are both Banach spaces, T^{-1} is bounded.

Note. The Bounded Inverse Theorem follows from Theorem 3.6 with $R(T)$ replaced with Y.

Revised: 6/14/2021