3.5. Closed Linear Operators

Note. This section gives another property of a linear operator which implies boundedness (and hence continuity, by Theorem 2.6).

Definition. Given $T : X \to Y$, the graph of $T$, denoted $G_T$, is the set $\{(x, y) \mid x \in X, y \in Y, y = Tx\}$, a subset of $X \times Y$.

Note. In this section, for $X$ and $Y$ normed linear spaces, we view $X \times Y$ as having the sup norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. This allows us to put a topology on $X \times Y$.

Definition. A linear operator $T$ is closed if its graph $G_T$ is closed in $X \times Y$ with respect to the sup norm.

Theorem 3.7. If $T \in \mathcal{L}(X, Y)$ is injective (one to one) and closed, then $T^{-1}$ is closed.

Lemma. If $T \in \mathcal{L}(X, Y)$, where $X$ and $Y$ are Banach spaces, is bounded then $T$ is closed.
Example 3.8. Let $X = C[0, 1]$ with the sup norm and $Y = \{f \in C[0, 1] \mid f(0) = 0\}$ with the sup norm. Define $T : X \to Y$ as $(Tf)(t) = \int_0^t f(s) \, ds$. Then $T$ is injective (one to one) and has as its inverse $T^{-1} = D$ (differentiation). $T$ is bounded (see Exercise 2.16) and so is closed by “Lemma.” So by Theorem 3.7, $T^{-1}$ is closed.

Theorem 3.9. Closed Graph Theorem.

If $T \in \mathcal{L}(X, Y)$ where $X$ and $Y$ are Banach spaces, then $T$ is closed if and only if it is bounded.