3.5. Closed Linear Operators

Note. This section gives another property of a linear operator which implies boundedness (and hence continuity, by Theorem 2.6).

Definition. Given $T : X \to Y$, the *graph* of T, denoted G_T, is the set $\{(x, y) \mid x \in X, y \in Y, y = Tx\}$, a subset of $X \times Y$.

Note. In this section, for X and Y normed linear spaces, we view $X \times Y$ as having the sup norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. This allows us to put a topology on $X \times Y$.

Definition. A linear operator T is *closed* if its graph G_T is closed in $X \times Y$ with respect to the sup norm.

Theorem 3.7. If $T \in \mathcal{L}(X, Y)$ is injective (one to one) and closed, then T^{-1} is closed.

Lemma 3.5.A. If $T \in \mathcal{L}(X, Y)$, where X and Y are Banach spaces, is bounded then T is closed.

Example 3.8. Let $X = C[0,1]$ with the sup norm and $Y = \{f \in C[0,1] \mid f(0) = 0\}$ with the sup norm. Define $T : X \to Y$ as $(Tf)(t) = \int_0^t f(s)\,ds$. Then T is injective (one to one) and has as its inverse $T^{-1} = D$ (differentiation). T is bounded (see Exercise 2.16) and so is closed by Lemma 3.5.A. So by Theorem 3.7, T^{-1} is closed.

Theorem 3.9. Closed Graph Theorem.

If $T \in \mathcal{L}(X,Y)$ where X and Y are Banach spaces, then T is closed if and only if it is bounded.

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