3.6. Uniform Boundedness Principle

Note. This section deals with showing that a subset $\mathcal{A} \subseteq \mathcal{B}(X, Y)$ is bounded. Notice that set $\mathcal{A}$ is a set of bounded linear operators. Just because each element of $\mathcal{A}$ is bounded, that does not mean that set $\mathcal{A}$ itself is necessarily bounded (consider $\mathbb{N}$). As the text mentions, boundedness of set $\mathcal{A}$ is often called "uniform boundedness."

Definition. Set $\mathcal{A} \subseteq \mathcal{B}(X, Y)$ is *pointwise bounded* if for each $x \in X$ the set $\mathcal{O}_x = \{Tx \mid T \in \mathcal{A}\}$ is bounded in Y.

Note. We can think of $\mathcal{O}_x$ as sort of an x-cross section of set $\mathcal{A}$. We might think of $TX = \{Tx \mid x \in X\}$ as a sort of y-cross section, although it is T that is constant in $\mathcal{A}$ and not $y \in Y$ that is held constant. With this interpretation, the following result simply says that if all x-cross sections are bounded and all y-cross sections are bounded (or T-cross sections if you like—this follows, of course, from the fact that each T if bounded) then set $\mathcal{A}$ is bounded.

Theorem 3.10. Uniform Boundedness Principle.

If X is complete, then a pointwise bounded subset $\mathcal{A}$ of $\mathcal{B}(X, Y)$ is bounded.

Note. Sometimes the Uniform Boundedness Principle is called the Banach-Steinhaus Theorem. The following application of the Uniform Boundedness Principle shows that $\mathcal{B}(X, Y)$ is closed under pointwise limits.

Theorem 3.11. Suppose that (T_n) is a pointwise convergent sequence of bounded linear operators from Banach space X to normed linear space Y. That is, for each $x \in X$ the sequence $(T_n x)$ converges to an element $Tx \in Y$. Then T is linear and bounded.

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