4.2. Semi-Inner Products

Note. In this section, we define inner products and establish some results that have very geometric interpretations.

Definition. A *semi-inner product* on a complex linear space X is a mapping from $X \times X$ to $\mathbb{C}$, denoted $\langle x, y \rangle$ for $(x, y) \in X \times X$, where for all $x, y, z \in X$ and for all $\alpha \in \mathbb{C}$ we have:

(a) Linearity in the first variable:

$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$\langle \alpha x, z \rangle = \alpha \langle x, z \rangle.$$

(b) Conjugate symmetry: $\langle y, x \rangle = \overline{\langle x, y \rangle}$.

(c) Positivity: $\langle x, x \rangle \geq 0$ (implying that $\langle x, x \rangle \in \mathbb{R}$).

If, in addition, $\langle x, x \rangle = 0$ implies $x = 0$, then $\langle \cdot, \cdot \rangle$ is an *inner product*. A linear space with a (semi) inner product is a (*semi*) *inner product space*.

Note. Property (a) can be summarized as

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle.$$

This combines with (b) to give "conjugate linearity" in the second variable:

$$\langle z, \alpha x + \beta y \rangle = \overline{\alpha} \langle z, x \rangle + \overline{\beta} \langle z, y \rangle.$$

Definition. For $\langle \cdot, \cdot \rangle$ a semi-inner product, define the (induced) semi-norm $\|x\| = \sqrt{\langle x, x \rangle}$.

Note. We easily have

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x \rangle} = \sqrt{\alpha \overline{\alpha} \langle x, x \rangle} = |\alpha| \|x\|.$$

Also, if $\langle \cdot, \cdot \rangle$ is an inner product, then $\|x\| = 0$ if and only if $x = 0$. A "semi-norm" $\| \cdot \|$ satisfies all the properties of a norm *except* the property "$\|x\| = 0$ implies $x = 0$." That is, there may be nonzero vectors with semi-norm 0. See Exercise 2.6 for some properties of semi-norms. In the rest of this section, if we refer to a semi-inner product space then it is understood that the symbols "$\| \cdot \|$" represent the semi-norm induced by the semi-inner product.

Example. For $X = \mathbb{C}^n$, an inner product is

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = \sum_{k=1}^{n} x_k \overline{y_k}.$$

Not surprisingly, for $f, g \in L^2$, an inner product on L^2 is given by $\langle f, g \rangle = \int_\Omega f \overline{g}$, as we will see.

Lemma 4.2. Basic Identity.

Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y \rangle.$$

Theorem 4.3. Cauchy-Schwartz Inequality.

Given a semi-inner product on X, for all $x, y \in X$ we have $|\langle x, y\rangle| \le \|x\|\|y\|$. If $\langle \cdot, \cdot \rangle$ is an inner product, then equality holds if and only if x is a scalar multiple of y (that is, x and y are linearly dependent).

Note. As you've seen before, the Cauchy-Schwartz Inequality is used to prove the Triangle Inequality.

Theorem 4.4. Triangle Inequality.

Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have $\|x + y\| \le \|x\| + \|y\|$.

Note. With the Triangle Inequality established, we see that if $\langle \cdot, \cdot \rangle$ is a (semi) inner product, then $\| \cdot \|$ is a (semi) norm.

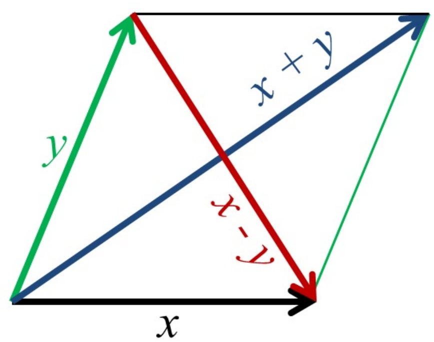
Note. We have seen many normed linear spaces. In the case of $\mathbb{R}^n$, $\mathbb{C}^n$, L^2, and ℓ^2, the (standard Euclidean) norms are "induced" by an inner product. However, these are special cases. The following result can be used to show that, among the L^p spaces, only for $p = 2$ is the norm induced by an inner product.

Proposition 4.5. Parallelogram Law.

Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Note. The reason this is called the Parallelogram law is that, in a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the lengths of the four edges:



Notice that the contrapositive of the Parallelogram law can be read as: "If the semi-norm in a normed linear space does not satisfy the Parallelogram Law, then the semi-norm is not induced by a semi-inner product." This can be used to show that the ℓ^p spaces are not inner product spaces for $1 \leq p < 2$ and $2 < p \leq \infty$. See Exercise 4.2.

Note. We know that if $\langle \cdot, \cdot \rangle$ is an inner product, then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm. The following result gives the inner product in terms of the norm induced by it.

Proposition 4.7. Polarization Identity.

Let X be a semi-inner product space. Then the semi-norm induced by the semi-inner product satisfies: for all $x, y \in X$, we have

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

Note. The following result tells us when a norm is induced by an inner product.

Theorem 4.8. If $\|\cdot\|$ is a norm on a (complex) linear space X satisfying the parallelogram law, then the norm is induced by an inner product.

Proof. Homework. See the text for hints.

Theorem 4.9. Continuity of Inner Product.

In any semi-inner product space, if the sequences $(x_n) \to x$ and $(y_n) \to y$, then $(\langle x_n, y_n \rangle) \to \langle x, y \rangle$.

Definition. A *Hilbert space* is a complete inner product space.

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