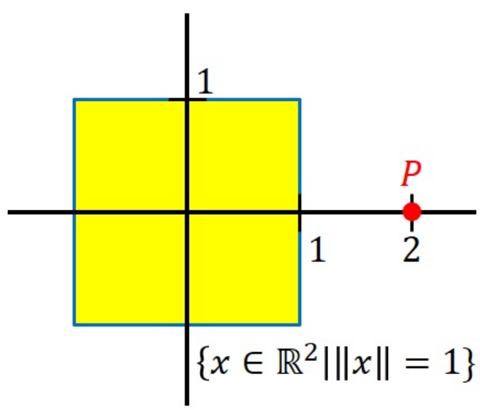
4.3. Nearest Points and Convexity

Note. As the title suggests, this section is about (in a Hilbert space) finding the closest point in a set to a given point. Recall that the distance from a point x to a set Y in a normed linear space is $d(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, so we would not in general expect there to be a "nearest point."

Definition. A set K in a linear space is *convex* if for all $x, y \in K$ and any scalar $\alpha \in [0, 1]$, we have $\alpha x + (1 - \alpha)y \in K$.

Note. Geometrically, a set K is convex when $x, y \in K$ implies that all points on a line connecting x and y are in K. By induction, if $x_1, x_2, \ldots, x_n \in K$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 1]$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then the point $\sum_{k=1}^{n} \alpha_k x_k \in K$.

Note. We are interested in conditions under which the "nearest point" is unique. Consider the point $P = (2, 0)$ and the set $\{x \mid \|x\| \leq 1\}$ in $\mathbb{R}^2$ under the sup norm. The set is convex, but any point in the closed unit ball with first coordinate equal to 1 is a point a distance 1 from P. So there is not a unique nearest point. The real problem is that the boundary of the set is "flat."



Definition. A normed linear space X is *strictly convex* if for any two distinct unit vectors x and y, we have $\|(x+y)/2\| < 1$.

Note. This definition implies that if x and y are boundary points on the unit ball, then the midpoint of x and y is not a boundary point. That is, the boundary contains no line segments. The boundary is, as the text says, "round."

Proposition 4.10. Suppose X is strictly convex. For any point x and convex set K, there is at most one point in K that is nearest to x.

Definition. A normed linear space is *uniformly convex* if for all $\varepsilon > 0$, there is $\delta > 0$ such that for $x, y \in \overline{B}(1) = \overline{B}(0;1)$ we have

$$\left\|\frac{1}{2}(x+y)\right\| > 1 - \delta \text{ implies } \|x-y\| < \varepsilon.$$

Lemma 4.3.A. If a normed linear space is uniformly convex, then it is strictly convex.

Example 4.11. Any Hilbert space is uniformly convex! Let $x, y \in \overline{B}(1)$ and $z = (x+y)/2$. By the Parallelogram Law (Proposition 4.5),

$$\|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x+y\|^2 \le 4 - (2\|z\|)^2 = 4(1 - \|z\|^2).$$

So if $\varepsilon \in (0,2)$ then let $\delta = 1 - \sqrt{1 - \varepsilon^2/4}$, in which case $(1-\delta)^2 = 1 - \varepsilon^2/4$ and so $\left\|\frac{1}{2}(x+y)\right\| = \|z\| > 1-\delta$ implies

$$\left\|\frac{1}{2}(x+y)\right\|^2 = \|z\|^2 > (1-\delta)^2 = 1 - \frac{\varepsilon^2}{4},$$

or

$$-\|z\|^2 < -(1-\delta)^2 = \frac{\varepsilon^2}{4} - 1,$$

or $4(1-\|z\|^2) < \varepsilon^2$. This implies that $\|x-y\|^2 \le 4(1-\|z\|^2) < \varepsilon^2$, or $\|x-y\| < \varepsilon$. If $\varepsilon \ge 2$, then any $\delta > 0$ works to show uniform convexity since $\|x-y\| \le \|x\|+\|y\| \le 2 \le \varepsilon$.

Note. The text states that the L^p spaces with $1 < p < \infty$ are uniformly convex spaces. The text references *A Short Course on Banach Spaces* by N.L. Carothers, Cambridge University Press (2005). For $p \ge 2$, this is an exercise (pages 87 and 88) in Reed and Simon's *Functional Analysis I*, Academic Press (1980).

Theorem 4.12. Suppose X is a uniformly convex Banach space. For any point x and a nonempty closed convex set K, there is a nearest point to x in K.

Note. A subspace of a Banach space is a convex set. Next, we let M be a closed subspace of a Hilbert space and define projections of x onto M as the point in M nearest to x. This projection idea will lead us into the Gram-Schmidt process and a discussion of orthonormal bases.

Definition. For M a closed subspace of a Hilbert space, define the *projection of x onto M* as the point in M nearest to x, denoted $P_M(x)$.