4.4. Orthogonality

Note. This section is **awesome**! It is very geometric and shows that much of the geometry of $\mathbb{R}^n$ holds in Hilbert spaces.

Definition. Elements x and y of a Hilbert space H are *orthogonal* if $\langle x, y\rangle = 0$, denoted $x \perp y$. For any subset S of a Hilbert space, denote

$$S^\perp = \{x \in H \mid \langle x, s\rangle = 0 \text{ for all } s \in S\}.$$

$S^\perp$ is called the *perp set* of set S.

Proposition 4.13. Suppose S is a subset of a Hilbert space H and suppose S is closed under scalar multiplication (i.e., $y \in S$ and $\alpha \in \mathbb{C}$ implies $\alpha y \in S$). Then

$$S^\perp = \{x \in H \mid d(x, S) = \|x\|\}.$$

Note. By Exercise 4.A, for any set S in a Hilbert space H, the set $S^\perp$ is a closed subspace of H.

Theorem 4.14. Projection Theorem.

Let M be a closed subspace of a Hilbert space H. Then:

(a) $M \cap M^\perp = \{0\}$.

(b) Any $z \in H$ can be written uniquely as the sum of an element of M and an element of $M^\perp$. Specifically, $z = P_M(z) + P_{M^\perp}(z)$.

(c) $M^{\perp\perp} = M$.

(d) H is isometric to $M \oplus M^\perp$ where the direct sum is equipped with the ℓ^2 norm.

Definition. If $\{x_k \mid k \in K\}$ is an orthonormal set for which the closed linear span (where the linear span is based on finite linear combinations) of set $\{x_k \mid k \in K\}$ is all of Hilbert space H, then $\{x_k \mid k \in K\}$ is an *orthonormal basis* of H.

Note. We now step aside and consider supplemental notes from *Real Analysis with an Introduction to Wavelets and Applications*, Don Hong, Jianzhong Wang, and Robert Gardner, Elsevier Academic Press (2005). The appropriate notes are from Sections 5.1, "Groups, Fields, and Vector Spaces," and 5.4, "Projections and Hilbert Space Isomorphisms." To briefly summarize:

Theorem 5.1.4. Let $\langle V, \mathbb{F}\rangle$ be a vector space. Then there exists a set of vectors $B \subset V$ such that (1) B is linearly independent and (2) for any $\mathbf{v} \in V$ there exists finite sets $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subset B$ and $\{f_1, f_2, \ldots, f_n\}$ such that $\mathbf{v} = f_1\mathbf{b}_1 + f_2\mathbf{b}_2 + \cdots + f_n\mathbf{b}_n$. That is, B is a Hamel basis for $\langle V, \mathbb{F}\rangle$.

Exercise 5.1.3. If B_1 and B_2 are Hamel bases for a given infinite dimensional vector space, then B_1 and B_2 are of the same cardinality.

Theorem 5.4.4. A Hilbert space with a Schauder basis has an orthonormal basis. (This is a consequence of the Gram-Schmidt process.)

Theorem 5.4.8. A Hilbert space with scalar field $\mathbb{R}$ or $\mathbb{C}$ is separable if and only if it has a countable orthonormal basis.

Theorem 5.4.9. Fundamental Theorem of Infinite Dimensional Vector Spaces. Let H be a Hilbert space with a countable infinite orthonormal basis. Then H is isomorphic to ℓ^2.

Note. We now return to Promoslow…

Theorem 4.17. Properties of Orthonormal Sets.

Let $\{x_k \mid k \in K\}$ be an orthonormal set in a Hilbert space H, and let M be the closed linear span of the elements of this sequence. Then:

(a) Given any sequence $\alpha = (\alpha_i)$ in $\mathbb{C}$, $\sum_{i=1}^{\infty} \alpha_i x_i$ converges to some element $x \in H$ if and only if $\alpha \in \ell^2$, and in this case $\|x\| = \|\alpha\|_2$.

(b) For any $z \in H$, $P_M(z) = \sum_{i=1}^{\infty} \langle z, x_i \rangle x_i$.

Theorem 4.18. Every Hilbert space has an orthonormal basis.

Note. Before we give a proof of Theorem 4.18, we review several set theoretic concepts. These can be found in Section 5.1 of Hong, Wang, and Gardner.

Definition. For set X, any subset of $X \times X$ is a *binary relation* (or simply *relation*) on X. A relation R on X is *reflexive* if for all $x \in X$, $(x, x) \in R$. R is *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$. R is *transitive* if $(x, y), (y, z) \in R$ implies $(x, z) \in R$.

Definition. A relation R on X is an *equivalence relation* if it is reflexive, symmetric, and transitive. A relation is *antisymmetric* if $(x, y) \in R$ and $(y, x) \in R$ implies that $x = y$. A relation is a *partial ordering* if it is reflexive, antisymmetric and transitive, and is denoted $\leq$. R is a *total ordering* if R is a partial ordering and for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

Definition. If set X is partially ordered by $\leq$, a *maximal* (or *minimal*) element of X is $x \in X$ such that $x \leq y$ (or $y \leq x$) implies $y = x$. If $E \subset X$, an *upper bound* for E is an element $x \in X$ such that $y \leq x$ for all $y \in E$.

Note. The following result is equivalent to the Axiom of Choice.

Zorn's Lemma. If X is a partially ordered set and every totally ordered subset of X has an upper bound, then X has a maximal element.

Note. We are now ready for the proof of Theorem 4.18.

Note. As in the proof that every vector space has a basis (see Theorem 5.1.4 of Hong, Wang, Gardner), Theorem 4.18 uses Zorn's Lemma. Zorn's Lemma guarantees the existence of an orthonormal basis of H without actually showing how to construct such a set. This means (as is also the case for a basis of a general vector space) that we have no idea what is in the orthonormal basis. This makes Theorem 4.18 useless for any kind of application. So we do not further study general Hilbert spaces, but only those with a *countable* orthonormal (Schauder) basis. We see in the Hong, Wang, Gardner supplements that this is equivalent to considering separable Hilbert spaces (see Theorem 5.4.8 of Hong, Wang, Gardner).

Theorem 4.19. Any separable infinite dimensional Hilbert space is isometric to ℓ^2.

Note. Theorem 4.19 is just a restatement of the "Fundamental Theorem of Infinite Dimensional Vector Spaces" (a title introduced in the Hong, Wang, and Gardner text).

Theorem 4.20. Gram-Schmidt Orthogonalization Process.
Given a linearly independent sequence (y_k) in H, there is an orthonormal sequence (x_k) such that, for any $n \in \mathbb{N}$, $\text{span}\{x_1, x_2, \ldots, x_n\} = \text{span}\{y_1, y_2, \ldots, y_n\}$.

Note. In fact, the spirit of Theorem 4.20 holds in infinite dimensional spaces, and the Gram-Schmidt Process can be used to create an orthonormal (Schauder) basis in an infinite dimensional Hilbert space with a countable basis (this is Theorem 5.4.4 of Hong, Wang, and Gardner).

Theorem 4.21. If S is an orthonormal set in any separable Hilbert space H, then S is either finite or countably infinite.

Note. Theorem 4.21 is also a consequence of Theorem 5.4.8 from Hong, Wang, and Gardner.

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