4.5. Linear Functionals on Hilbert Spaces

Note. In this section, we classify the bounded linear functionals on a Hilbert space. In fact, the bounded linear functionals themselves form a Hilbert space.

Definition. Let X be a normed linear space over scalar field $\mathbb{F}$. Then a *bounded linear functional* is an element of $\mathcal{B}(X, \mathbb{F})$.

Note. We denote $\mathcal{B}(X, \mathbb{F})$ as X^*. We will see that for H a Hilbert space, H^* itself is a Hilbert space called the *dual space* of H. The following is a "representation theorem." It gives a classification of bounded linear functionals in terms of inner products. The analogous result in the classical Banach spaces (the L^p spaces) is called the *Riesz Representation Theorem.*

Theorem 4.22. For any z in a Hilbert space H, the functional ψ_z defined by $\psi_z(x) = \langle x, z \rangle$ is in $H^* = \mathcal{B}(H, \mathbb{F})$. Conversely, given any $f \in H^*$, there is a unique $z \in H$ such that $f = \psi_z$.

Note. Theorem 4.22 tells us that every element $z \in H$ corresponds to a bounded linear functional $\psi_z \in H^*$ of the form $\psi_z(x) = \langle x, z\rangle$, and conversely every bounded linear functional in H^* corresponds to some ψ_z. This ability to represent bounded linear functionals in terms of inner products is often called the Riesz Representation Theorem.

Definition. Define J_H as the mapping from H to H^* that sends $z \in H$ to $\psi_z \in H^*$.

Note. By Theorem 4.22, $J_H : H \to H^*$ is bijective (one to one and onto) and an isometry (since $\|\psi_z\| = \|z\|$ as shown in the proof of Theorem 4.22). However, J_H is not linear since

$$J_H(iy)(x) = \psi_{iy}(x) = \langle x, iy\rangle = -i\langle x, y\rangle = -i\psi_y(x) = -iJ_H(y)(x) \neq iJ_H(y)(x).$$

That is, J_H is conjugate linear.

Corollary 4.23. For any Hilbert space H, the dual H^* is a Hilbert space with the same dimension as H.

Idea of Proof. For $g, f \in H^*$, we have by Theorem 4.22 that $f = \psi_x$ and $g = \psi_y$ for some $x, y \in H$. Define $\langle f, g\rangle = \langle y, x\rangle$. Then this is an inner product on H^* and the norm on H^* is induced by this inner product. By continuity of the inner product (Theorem 4.9), Cauchy sequences in H^* correspond to Cauchy sequences in H (and so do the limits). Suppose $z_1, z_2 \in H$ and $\langle z_1, z_2\rangle = 0$. Then in H^*,

$$\langle J_H(z_1), J_H(z_2)\rangle = \langle \psi_{z_1}, \psi_{z_2}\rangle$$

$$= \langle z_2, z_1\rangle (\text{by definition of inner product in } H^*) = \overline{\langle z_1, z_2\rangle} = \overline{0} = 0.$$

So if (x_n) is an orthonormal basis in H, then $(J_H(x_n))$ is an orthonormal basis of

H^* (length is preserved since J_H is an isometry). So the dimension of H and the dimension of H^* are the same. ■

Definition. With the above notation, if (x_n) is an orthonormal basis of H then $(J_H(x_n))$ is the orthonormal *dual basis* of H^*.

Note. Since every infinite dimensional separable Hilbert space is isomorphic to ℓ^2 (Theorem 4.19), then the dual space of any infinite dimensional separable Hilbert space H is itself. That is, $H = H^*$.

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