4.6. Linear Operators on Hilbert Spaces

Note. This section explores a number of different kinds of bounded linear transformations (or, equivalently, "operators") from Hilbert space H to itself. We define and describe the adjoint of a bounded linear transformation, self adjoint operators, normal operators, positive operators, and unitary operators. The text doesn't mention it, but these operators have applications to quantum mechanics. A nice reference for applications to quantum mechanics is *Introduction to Hilbert Spaces with Applications* by Lokenath Debnath and Piotr Mikusínski, 3rd Edition, Elsevier Academic Press (2005) (see Chapter 7: Mathematical Foundations of Quantum Mechanics).

Definition. A *sesquilinear form* on Hilbert space H is a function from $H \times H$ to $\mathbb{C}$ such that for all $x, y, z \in H$ and $\alpha \in \mathbb{C}$ we have

$$f(x+y,z) = f(x,z) + f(y,z), \;\; f(\alpha x, z) = \alpha f(x,z)$$

$$f(z,x+y) = f(z,x) + f(z,y), \;\; f(x,\alpha z) = \overline{\alpha} f(x,z).$$

Note. The term "sesqui" means one-and-a-half and is used since f is linear in the first entry and half linear in the second. Of course, an inner product is an example of a sequilinear form.

Definition. For any sesquilinear form f, the function $q : H \to \mathbb{C}$ defined as $q(x) = f(x,x)$ is a *quadratic form* associated with f.

Definition. For sesquilinear form f on Hilbert space H, define the *norm* of f as

$$\|f\| = \sup\{|f(x,y)| \mid x, y \in H, \|x\| = \|y\| = 1\}.$$

If $\|f\| < \infty$, then f is *bounded.*

Note. Similar to Theorem 4.22 which gave a representation of bounded linear functionals, the following gives a representation of bounded sesquilinear forms.

Theorem 4.24. Given any $T \in \mathcal{B}(H)$ (the set of bounded linear transformations from H to itself), the function f_T defined by $f_T(x,y) = \langle Tx, y\rangle$ is a sesquilinear form with norm equal to $\|T\|$. Conversely, given any bounded sesquilinear form f, there is a unique $T \in \mathcal{B}(H)$ such that $f = f_T$.

Corollary 4.25. If $\langle Tx, x\rangle = \langle Sx, x\rangle$ for all $x \in H$ where H is a Hilbert space with complex scalars, then $T = S$.

Note. Corollary 4.25 follows from Theorem 4.24 based on the uniqueness claim. However, notice that Corollary 4.25 only holds when $\mathbb{F} = \mathbb{C}$. For example, for $T(x_1, x_2) = (-x_2, x_1)$ on $\mathbb{R}^2$, then $\langle Tx, x\rangle = 0 = \langle 0x, x\rangle$ for all $x \in \mathbb{R}^2$, but $T \neq 0$.

Definition. Given any $T \in \mathcal{B}(H)$ (a bounded linear transformation from H to itself), its *adjoint operator*, denoted T^*, is the unique element of $\mathcal{B}(H)$ (by Theorem 4.24) associated with the sesquilinear form $f(x,y) = \langle x, Ty\rangle$.

Note. By Theorem 4.24, T^* satisfies $f(x, y) = \langle T^* x, y\rangle$, so we have $\langle Tx, y\rangle = \langle x, T^* y\rangle$.

Note. In finite dimensions, we know that T and T^* are represented by matrices. Let $\{x_1, x_2, \ldots, x_n\}$ be an ordered basis for H. Then the matrix $A_T = (a_{ij})$ which represents T is (see page 7) $a_{ij} = \langle Tx_j, x_i\rangle$ (apply T to the ordered jth basis element to get the jth column and then project that onto the ith basis element to get a_{ij}). Similarly, the matrix representing T^* is $A_{T^*} = (a^*_{ij})$ where

$$a^*_{ij} = \langle T^* x_j, x_i\rangle = \langle x_j, Tx_i\rangle = \overline{\langle Tx_i, x_j\rangle} = \overline{a}_{ji}.$$

So the relationship between A_T and A_{T^*} is that of "conjugate transpose."

Theorem 4.26. Properties of Hilbert Space Adjoints.
Given $S, Y \in \mathcal{B}(H)$ and $\alpha \in \mathbb{C}$:

(a) $(S + T)^* = S^* + T^*$

(b) $(\alpha T)^* = \overline{\alpha} T^*$

(c) $(ST)^* = T^* S^*$

(d) $\|T^*\| = \|T\|$

(e) $T^{**} = T$

(f) $\|T^* T\| = \|T\|^2$.

Note. The following relates the nullspace of a bounded linear operator from a Hilbert space to itself to the range of the adjoint (and vice versa) in a rather geometric way.

Proposition 4.27. For all $T \in \mathcal{B}(H)$ (the set of bounded linear transformations from H to itself):

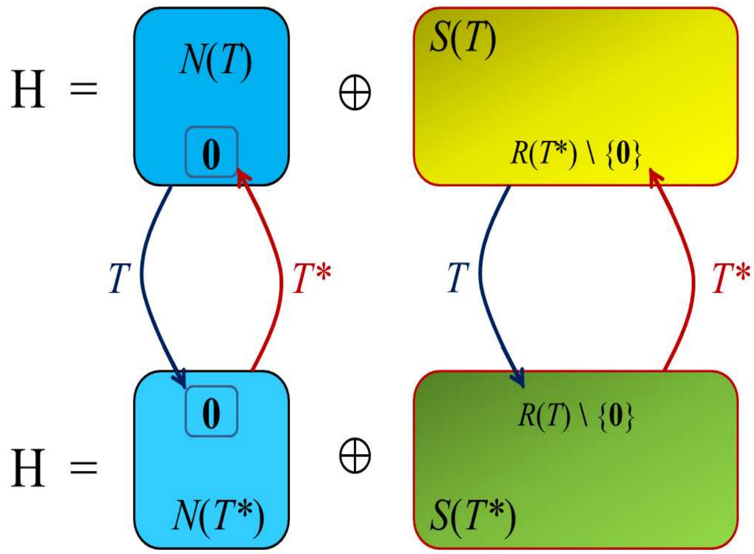
(a) $N(T^*) = R(T)^\perp$

(b) $N(T)^\perp = \overline{R(T^*)}$.

Note. Recall that, in general, the "support" of a function is where the function is nonzero. This is consistent with the following (geometric) definition.

Definition. The *support* of $T \in \mathcal{B}(H)$, denoted $S(T)$, is defined as $N(T)^\perp$ (the perp space of the nullspace).

Note. By Proposition 4.27, the support of T^* equals the closed linear span of T: $S(T^*) = \overline{R(T)}$. Similarly, $S(T) = \overline{R(T^*)}$. So $S(T)$ and $S(T^*)$ are closed subspaces of H. So, by Theorem 4.14(d), we have that $H = N(T) \oplus S(T)$ and $H = N(T^*) \oplus S(T^*)$. We can view T and T^* as mapping these two copies of H into each other. The publishers thought this a big deal and put the figure illustrating this (Figure 4.4) on the cover of the text. This figure illustrates that T maps $N(T)$ to 0 and $S(T)$ to a subset of $S(T^*)$ (since $\overline{R(T)} = S(T^*)$), and conversely with T replaced by T^*. Notice that these mappings concern the *parts* of H in the direct sum decomposition, and not mappings of the *elements* of H themselves. We might illustrate this as follows:



Note. We now study several classes of operators (i.e., elements of $\mathcal{B}(H)$).

Definition. An element $T \in \mathcal{B}(H)$ is *normal* if $TT^* = T^*T$.

Example. Define T on H as $Tx = ix$. Then

$$\langle Tx, y\rangle = \langle ix, y\rangle = i\langle x, y\rangle = \langle x, -iy\rangle = \langle x, T^*y\rangle,$$

so $T^*x = -ix$. Then $TT^*(x) = T(-ix) = i(-ix) = x$ and $T^*T(x) = T^*(ix) = -i(ix) = x$, so $T^*T = TT^*$ and T is normal.

Proposition 4.30. T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.

Note. If T is normal, then $N(T) = N(T^*)$ (since $\|Tx\| = 0$ if and only if $\|T^*x\| = 0$) and so the supports are the same as well, $S(T) = S(T^*)$. So from the diagram above and Proposition 4.27, we have that T and T^* both map $S(T) = S(T^*)$ to a dense subset of $S(T) = S(T^*)$. This is used in the proof of Proposition 4.33 below.

Definition. $T \in \mathcal{B}(H)$ is *self adjoint* if $T = T^*$.

Note. Recall that in finite dimensions, the matrix representation of T and T^* are conjugate transposes of each other. If $\mathbb{F} = \mathbb{R}$, then the conjugation does not play a role and the matrices are simply transposes of each other. So, in finite dimensional real space $\mathbb{R}^n$, T is self adjoint if its matrix representation is symmetric.

Proposition 4.31. T is self-adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in H$.

Definition. An element $T \in \mathcal{B}(H)$ is *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Note. Since $\langle Tx, x\rangle \geq 0$ implies $\langle Tx, x\rangle$ is real, then all positive operators are also self adjoint by Proposition 4.31. Also, for any $T \in \mathcal{B}(H)$, the operator TT^* is positive since $\langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2 \geq 0$. The following result gives a "funny" property of positive operators—they have square roots!

Theorem 4.32. Given any positive operator T, there is a unique positive operator A such that $A^2 = T$. Moreover, A commutes with any operator that commutes with T. A is called the *square root* of T, denoted $A = T^{1/2}$.

Note. The proof of Theorem 4.32 is given in Chapter 8.

Definition. An element $P \in \mathcal{B}(H)$ is a *projection* if $P = P^*$ and $P^2 = P$.

Note. Projections are positive since

$$\langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, P^*x\rangle = \langle Px, Px\rangle = \|Px\|^2 \geq 0.$$

Proposition 4.33. An element $P \in \mathcal{B}(H)$ is a projection if and only if there is a closed subspace M of H such that $P = P_M$ (the projection onto M, see page 79).

Note. The text makes the following claims and describes them as "relating a geometric statement about subspaces with an algebraic one about projection operators." Let M and N be closed subspaces of Hilbert space H.

(a) $P_M P_N = 0$ if and only if M and N are orthogonal, and in this case, $P_M + P_N$ is a projection onto the closed subspace spanned by $M \cup N$.

(b) $P_M P_N = P_M$ if and only if $M \subseteq N$, and in this case, $P_N - P_M$ is the projection onto the subspace $N \cap M^\perp$.

(c) $P_M P_N$ is a projection if and only if $P_M P_N = P_N P_M$, and in this case, it is the projection onto the closed subspace $K + M \cap N$. This will occur if and only if the subspaces $M \cap K^\perp$ and $N \cap K^\perp$ are orthogonal.

Definition. An element $U \in \mathcal{B}(H)$ is *unitary* if $U^*U = UU^* = I$ (the identity operator).

Example. We saw above that $Tx = ix$ is a unitary operator with $T^*x = -ix$.

Proposition 4.34. An element $U \in \mathcal{B}(H)$ is unitary if and only if it is a surjective (onto) isometry.

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