4.7. Order Relation on Self-Adjoint Operators

Note. Just as the positive set allows us to put an order on the reals, the existence of positive operators allows us to put a *partial* ordering on the set of self-adjoint operators.

Definition. Self-adjoint operators A and B satisfy the condition A *is less than or equal to* B, denoted $A \leq B$, if operator $B - A$ is positive.

Note. If T_1 and T_2 are positive (i.e., $\langle T_1x, x\rangle \geq 0$ and $\langle T_2x, x\rangle \geq 0$ for all $x \in H$), then

$$\langle (T_1 + T_2)x, x\rangle = \langle T_1x + T_2x, x\rangle = \langle T_1x, x\rangle + \langle T_2x, x\rangle \geq 0.$$

So the sum of two positive operators is positive. So, if $A \leq B$ and $B \leq C$, then $(B - A) + (C - B) = C - A \geq 0$ and so $A \leq C$. So transitivity holds. For all self-adjoint A, $A \leq A$ since $A - A$ is positive, and "$\leq$" is reflexive. If $A \leq B$ and $B \leq A$ then for all $x \in H$, $\langle (B - A)x, x\rangle \geq 0$ and $\langle (A - B)x, x\rangle \geq 0$. Notice $\langle (B - A)x, x\rangle + \langle (A - B)x, x\rangle = \langle 0, x\rangle = 0$ for all $x \in H$. So it must be that $\langle (B - A)x, x\rangle = 0$ and $\langle (A - B)x, x\rangle = 0$ for all $x \in H$. Therefore, $A - B = 0$ by Corollary 4.25, and $A = B$. Therefore $\leq$ is a partial ordering.

Proposition 4.38. Given two closed subspaces M and N, the projection P_M and P_N satisfy $P_M \leq P_N$ if and only if $M \subseteq N$.