5.2. Basic Version of Hahn-Banach Theorem

Note. In this section, we restrict ourselves to linear spaces of $\mathbb{R}$. We then take advantage of the fact that $\mathbb{R}$ has an ordering (unlike $\mathbb{C}$) to compare the values of linear functionals.

Definition. A *Minkowski functional* on a linear space X is a real-valued function p such that for all $x, y \in X$ we have:

(i) $p(x + y) \leq p(x) + p(y)$,

(ii) $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$.

Note. Royden and Fitzpatrick (in Section 14.2, The Hahn-Banach Theorem, of *Real Analysis*, 4th edition, Prentice Hall 2010) call property (i) "subadditivity" and property (ii) "positive homogeneity." So they use the terminology "positively homogeneous subadditive functional" instead of "Minkowski functional." They (and Reed and Simon in Chapter V, Locally Convex Spaces, of *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press 1980) use the term "Minkowski functional" for a specific functional which we will encounter in Section 5.5 as the "Minkowski functional of convex set K," denoted p_K (Royden and Fitzpatrick also use the term "gauge functional" for p_K).

Note. Norms (even seminorms) are examples of Minkowski functionals.

Theorem 5.1. Hahn-Banach Extension Theorem.

Suppose that p is a Minkowski functional on a real linear space X and f_0 is a linear functional defined on a subspace Y of X such that $f_0(y) \leq p(y)$ for all $y \in Y$. Then, f_0 has an extension to a linear functional f defined on X such that $f(y) = f_0(y)$ for all $y \in Y$ and $f(x) \leq p(x)$ for all $n \in X$.

Note. There is no boundedness condition on p, so there is no guarantee that f is bounded. However, in a sense, p acts as a type of bound on f (and if p *is* bounded, then so is f).

Note. The proof of the Hahn-Banach Extension Theorem follows two "stages." In Stage 1 we extend f_0 from subspace Y to a space one dimension greater than the dimension of y. This yields the result if the dimension of X is only a finite number greater than the dimension of Y. In Stage 2 we use Zorn's Lemma and the existence of a certain maximal element under a total ordering to get the extension in the event of infinite dimensional spaces. Now for the proof.

Note. We do not necessarily have uniqueness of the extension, as suggested by the choice of r in Stage 1 of the proof (unless $\sup A = \inf B$ in the proof).

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