5.3. Complex Version of the Hahn-Banach Theorem

Note. In this section, we consider linear spaces with complex scalars. Since there is no ordering on \( \mathbb{C} \), we cannot compare the values of linear functionals but we can compare the sizes (moduli) of linear functionals.

Note. We denote the set of all linear mappings from \( X \) to \( \mathbb{C} \) as \( X^\mathbb{C} \). (Be careful when reading the text, because their use of this notation looks a lot like \( X \) complement!)

**Proposition 5.2.** A function \( f : X \to \mathbb{C} \) is in \( X^\mathbb{C} \) (i.e., \( f \) is a complex valued linear functional) if and only if Re\((f)\) and Im\((f)\) are both linear real valued functionals on \( X \) and, for all \( x \in X \), Im\((f(x)) = -\text{Re}(f(ix))\).

**Theorem 5.3.** Complex Hahn-Banach Extension Theorem.
Suppose \( \| \cdot \| \) is a seminorm on a complex linear space \( X \) and that \( f_0 \) is a linear functional defined on a subspace \( Y \) of \( X \) such that \( |f_0(y)| \leq \|y\| \) for all \( y \in Y \). Then \( f_0 \) has an extension to a linear functional \( f \) on \( X \) such that \( |f(x)| \leq \|x\| \) for all \( x \in X \).

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