5.3. Complex Version of the Hahn-Banach Theorem

Note. In this section, we consider linear spaces with complex scalars. Since there is no ordering on $\mathbb{C}$, we cannot compare the *values* of linear functionals but we can compare the *sizes* (moduli) of linear functionals.

Note. We denote the set of all linear mappings from X to $\mathbb{C}$ as $X^{\mathbb{C}}$. (Be careful when reading the text, because their use of this notation looks a lot like X complement!)

Proposition 5.2. A function $f : X \to \mathbb{C}$ is in $X^{\mathbb{C}}$ (i.e., f is a complex valued linear functional) if and only if $\text{Re}(f)$ and $\text{Im}(f)$ are both linear real valued functionals on X and, for all $x \in X$, $\text{Im}(f(x)) = -\text{Re}(f(ix))$.

Theorem 5.3. Complex Hahn-Banach Extension Theorem.
Suppose $\|\cdot\|$ is a seminorm on a complex linear space X and that f_0 is a linear functional defined on a subspace Y of X such that $|f_0(y)| \leq \|y\|$ for all $y \in Y$. Then f_0 has an extension to a linear functional f on X such that $|f(x)| \leq \|x\|$ for all $x \in X$.

Revised: 5/20/2015