5.4. Application to Normed Linear Spaces

Note. We now consider a normed linear space with scalar field either $\mathbb{R}$ or $\mathbb{C}$. This section is very geometric and foreshadows the results of the next section.

Theorem 5.4. Normed Linear Space Version of Hahn-Banach Extension Theorem.
Suppose that f_0 is a bounded linear functional defined on a subspace Y of a normed linear space X. Then f_0 has an extension to a bounded linear functional f on X such that $\|f\| = \|f_0\|$.

Note. The text claims that the following result insures that there are many linear functionals on any normed linear space. However, the beauty of the result is that it gives us the liberty to define linear functionals with specific values at given points.

Corollary 5.5. Given any closed subspace Y of a normed linear space X and $x \notin Y$, there is a bounded linear functional f on X (i.e., $f \in X^*$) such that $f(Y) = 0$ and $f(x) = 1$.

Note. The following result is reminiscent of a result from topology concerning the separation of points. In fact, the function which is shown to exist in the result is said to "separate" points x and y.

Corollary 5.6. Given a normed linear space X and two points $x \neq y$, there is a bounded linear functional (i.e., $f \in X^*$) such that $f(x) \neq f(y)$.

Corollary 5.7. Consider linear space X with dual space X^*. The closed unit ball in X^* consists of all bounded linear functionals on X of functional norm less than or equal to 1. For any $x \in X$, we have

$$\|x\| = \sup\{|f(x)| \mid f \text{ is in the closed unit ball of } X^*\}.$$

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