5.5. Geometric Versions of Hahn-Banach Theorem

Note. We again need an ordering and so return t the setting of real linear spaces in this section. We define new topological ideas of internal/external/bounding points. Mimicing the idea of separating sets in $\mathbb{R}^2$ with lines, we give several separation theorems where sets are separated by hyperplanes.

Note. Recall that a *linear manifold* in a linear space X is a set of the form $\{x+Y\}$ where Y is a subspace of X.

Definition. A linear manifold for which Y is a maximal proper subspace of X (i.e., the only subspace of X properly containing Y is X itself) is a *hyperplane*.

Note. In $\mathbb{R}^n$, a hyperplane is a translation of $\mathbb{R}^{n-1}$.

Example. Let f be a nonzero linear functional with nullspace N. Then for any $x \in N$ we have

$$y = \left(y - \frac{f(y)}{f(x)}x\right) + \frac{f(y)}{f(x)}x$$

for all $y \in X$. Now

$$f\left(y - \frac{f(y)}{f(x)}x\right) = f(y) - \frac{f(y)}{f(x)}f(x) = 0,$$

so $y - \dfrac{f(y)}{f(x)}x \in N$ for any $x \in N$ and for all $y \in X$. So $y \in \text{span}(N \cup \{x\})$ for all $y \in X$. That is, $\text{span}(N \cup \{x\}) = X$ for any $x \notin N$. Therefore, N is a maximal proper subspace of X. Notice that $N = f^{-1}(\{0\})$. More generally, $f^{-1}(\{\alpha\}) = \{x \mid f(x) = \alpha\} = x + N$ where $x \in X$ satisfies $f(x) = \alpha$, is a hyperplane.

Note. In fact, every hyperplane is of the form $f^{-1}(\{\alpha\})$ for some linear functional and form $H = x + Y$ where Y is a maximal subspace. If H is not a subspace then $x \notin Y$ and since Y is maximal then $X = \text{span}(Y \cup \{x\})$. Define $f : X \to \mathbb{R}$ as $f(y + \alpha x) = \alpha$. Then f is linear and $f^{-1}(\{1\}) = \{x + Y \mid y \in Y\} = x + Y = H$. In the case that H is a subspace, choose any $x \in H$ and define $f(y + \alpha x) = \alpha$ and then $H = f^{-1}(\{0\})$.

Lemma 1. Let H be a hyperplane in X. If $H = f^{-1}(\{\alpha\})$ and $H = g^{-1}(\{\beta\})$ for some linear functionals f and g, then $f = \gamma g$ for some $\gamma \in \mathbb{R}$.

Definition. Given a hyperplane $H = f^{-1}(\{\alpha\})$, consider the sets $\{x \mid f(x) \leq \alpha\}$ and $\{x \mid f(x) \geq \alpha\}$. There are *half spaces determined by* H. If the inequalities are strict, the sets are called the *open* half spaces determined by H.

Note. Half spaces are well-defined—that is, independent of the linear functional f in the definition, as shown in Exercise 5.1.

Definition. Two subsets A and B of X are *separated* by hyperplane H if they lie in different half spaces determined by H. They are *strictly separated* if they lie in different open half spaces determined by H.

Note. Hyperplanes and half spaces are determined by linear functionals. In this section, we have not required the functionals to be bounded. Hyperplanes fall into two categories based on whether the linear functional on which they are based is bounded or not. Given $H = x + Y$ a hyperplane, continuity of addition and scalar multiplication (Theorem 2.3) imply $\overline{H} = x + \overline{Y}$ and $\overline{Y}$ is a linear subspace. Since Y is maximal and $Y \subseteq \overline{Y}$ then either $Y = \overline{Y}$ or $\overline{Y} = X$. If Y is closed then $Y = \overline{Y}$. If Y is not closed then $\overline{Y} = X$ and Y is dense in X. By Theorem 2.32, the linear functional f for which $H = N(f) = f^{-1}(\{0\})$ is closed if and only if f is bounded. So if f is bounded, hyperplane $f^{1}(\{\alpha\})$ is a closed subset of X, and if f is unbounded then hyperplane $f^{-1}(\{\alpha\})$ is a dense subset of X. In both cases, $f^{-1}(\{0\})$ is a maximal subspace of X, and H is a translation of this subspace.

Definition. For a subset A of a real linear space, a point $a \in A$ is *internal* to A if given any vector x there exists $r_x > 0$ such that $\{a + tx \mid 0 \le t \le r_x\} \subseteq A$. (Notice that we may without loss of generality assume x is a unit vector.)

Note. If a is an interior point of a set A in a normed linear space then, by definition, there is $r > 0$ such taht $B(a;r) \subset A$. So for any nonzero vector x we can choose $0 < r_x < r/\|x\|$ and then $a + tx \subset B(a;r) \subset A$ for $0 \le t \le r_x < r/\|x\|$. So a is an internal point of A. That is, interior points are always internal points. The converse does not hold in general, as shown in the following example.

Example 5.8. In $\mathbb{R}^2$ consider the set

$$A = \{(r, \theta) \mid r < 1 - \cos\theta\} \cup \{(r, \theta) \mid \theta = 0, 0 \leq 1 \leq 1\}.$$

Then 0 is internal to A, but 0 is not an interior point (as defined in Section 2.2). **Weird, eh!** This does not happen for convex sets in finite dimensions, as the following shows.

Note. The following proposition shows in $\mathbb{R}^n$ that internal points of a convex set are interior points

Proposition 5.9. For a convex set in $\mathbb{R}^n$, all internal points are interior.

Note. We denote the set of all internal points of set A as A^o. For set A from Example 5.8, we have $A^o = \{(r, \theta) \mid r < 1 - \cos\theta\} \cup \{0\}$ but $A^{oo} = \{(r, \theta) \mid r < 1 - \cos\theta\}$, so $A^o \neq A^{oo}$.

Proposition 5.10. If A is convex, then $A^{oo} = A^o$.

Definition. A point a is *external* to set A if it is internal to $A^c = X \setminus A$. A point that is neither internal to nor external to set A is a *bounding point* of set A.

Note. In a real normed linear space, any bounding point is a boundary point, but not necessarily conversely. In Example 5.8, 0 is a bound*ary* point and an internal point (so not a bound*ing* point).

Proposition 5.11. Given a Minkowski functional p, let $K_p = \{x \mid p(x) < 1\}$. Then K_p is convex and 0 is an internal point of K_p.

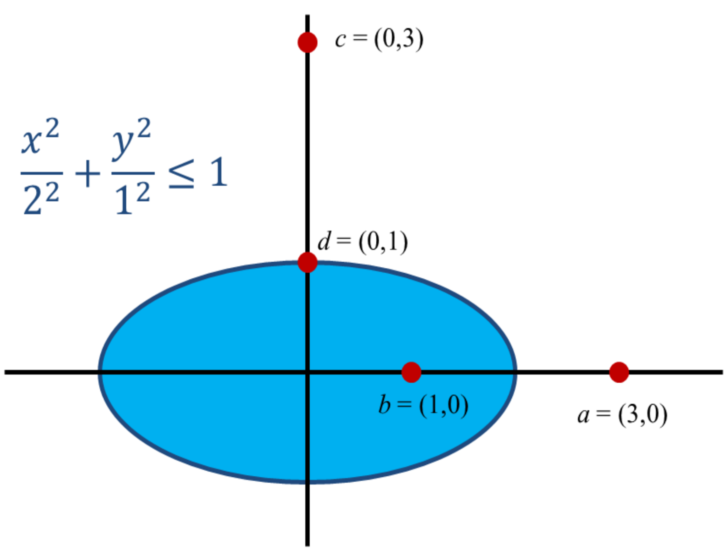
Note. Theorem 5.11 shows that a Minkowski functional can be used to construct a convex set with 0 as an internal point. We now show that the converse holds and that a convex set with 0 as an internal point can be used to construct a Minkowski functional.

Definition. Let K be a convex set in a real linear space X where 0 is an internal point of K. For all $x \in X$, define

$$p(x) = \inf\{t > 0 \mid x/t \in K\}.$$

We will see that p is a Minkowski functional which we denote p_K.

Note. The text illustrates the idea behind $p(x)$ using an ellipse. Consider the set K of points $(x, y) \in \mathbb{R}^2$ such that $\dfrac{x^2}{2^2} + \dfrac{y^2}{1^2} \leq 1$.



"We can view $p(x)$ as measuring the 'distance' of x from 0, but the unit of measurement varies from point to point. The unit of measurement for x is the line segment from 0 to the boundary of K in the direction of the point x." [Promislow, page 112] So the unit of measurement really varies by direction and not by location. Consider the points $a = (3, 0)$, $b = (1, 0)$, $c = (0, 3)$, and $d = (0, 1)$. We find that $p(3, 0) = 3/2$ (since $\frac{2}{3}(3, 0) = (2, 0)$ is on the boundary of K), $p(1, 0) = 1/2$ (since $2(1, 0) = (2, 0)$ is on the boundary of K), $p(0, 3) = 3$ (since $\frac{1}{3}(0, 3) = (0, 1)$ is on the boundary of K), and $p(0, 1) = 1$ (since $1(0, 1) = (0, 1)$ is on the boundary of K). We can interpret this as the unit of measure along the x-axis is 2, the unit of measure along the y-axis is 1, and in other directions the unit of measure is determined by the distance from 0 to the boundary of K.

Lemma 2. In a real normed linear space with a given convex set K which has 0 as an internal point. Define

$$p(x) = \inf\{t > 0 \mid x/t \in K\}.$$

If $p(x) < 1$ then $x \in K$. If $p(x) > 1$ then $x \notin K$.

Proof. If $x \in K$, then

$$p(x) = \inf\{t > 0 \mid x/t \in K\} \leq 1.$$

By the contrapositive of this statement, if $p(x) > 1$ then $x \notin K$.

If $x \notin K$ then for all $t > 1$ we have that $tx \notin K$ (we have $0 \in K$ by definition and *if* $tx \in K$ for $t > 1$, then $x \in K$ by convexity of K, a contradiction). So for $x \notin K$, if $x/t \in K$ then $1/t < 1$ and so $p(x) \geq 1$. By the contrapositive of this statement, we have that if $p(x) < 1$ then $x \in K$. ∎

Proposition 5.12. In a real normed linear space with a given convex set K which has 0 as an internal point. Define

$$p(x) = \inf\{t > 0 \mid x/t \in K\}.$$

Then:

(a) p is a Minkowski functional,

(b) $p(x) < 1$ if and only if x is an internal point of K,

(c) $p(x) = 1$ if and only if x is a bounding point of K, and

(d) $p(x) > 1$ if and only if x is an external point of K.

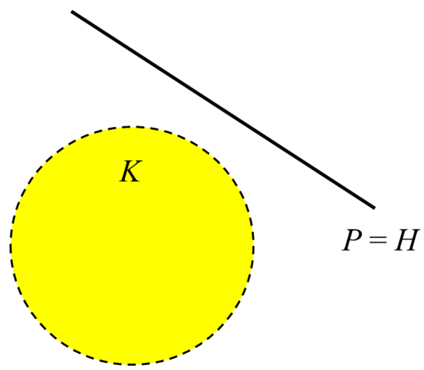
Definition. For convex set K with 0 as an internal point in a real normed linear space, the Minkowki functional $p(x) = \inf\{t > 0 \mid x/t \in K\}$ of Proposition 5.12 is called the *Minkowski functional of* K, denoted $p_K(x)$.

Note. Royden and Fitzpatrick (in Section 14.5, The Separation of Convex Sets and Mazur's Theorem, of *Real Analysis*, 4th edition, Prentice Hall 2010) call p_K the "gauge functional" for K. Reed and Simon (in Chapter V, Locally Convex Spaces, of *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press 1980,) use both of these terms.

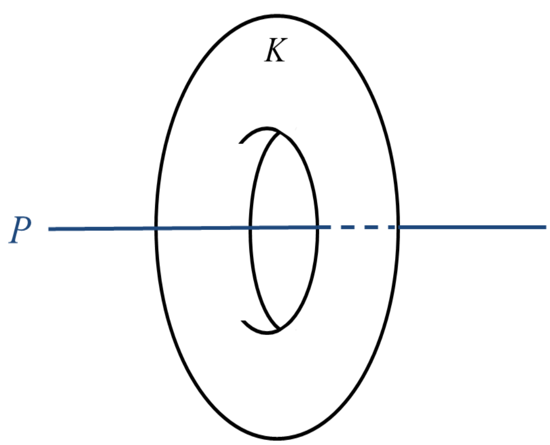
Proposition 5.13. Let K be a convex set which has some internal point and let f be a real valued linear functional on X. Then $f(K^o)$ is the interior of the interval $f(K)$.

Theorem 5.14. Geometric Hahn-Banach Extension Theorem.
We consider a real normed linear space. Let K be a convex set with an internal point, and let P be a linear manifold such that $P \cap K^o = \varnothing$. Then there is a hyperplane H containing P such that $H \cap K^o = \varnothing$.

Note. The text illustrates the above result in 3 dimensions on page 115. We can also illustrate it in 2 dimensions as:



We can't illustrate why convexity is necessary in 2 dimensions, but we can in 3 dimensions. Consider a torus in $\mathbb{R}^3$ with a line running through the hole. We cannot separate these with a plane:



Theorem 5.15. Hahn-Banach Separation Theorem.

We consider a real normed linear space. Let K and L be convex sets such that K has some internal point and $L \cap K^o = \varnothing$. Then there is a hyperplane separating K and L.

Note. The Hahn-Banach Separation Theorem deals with "separation" versus "strict separation." We will see strict separations in Theorem 5.17.

Definitions. If x is a bounding point of convex set K, then the hyperplane separating K and $L = \{x\}$ is called a *supporting hyperplane* to K.

Note. When you hear "supporting hyperplane" you might think "tangent plane." The following result is suggestive of the fact that in $\mathbb{R}^2$, a convex n-gon is the intersection of several half planes.

Theorem 5.16. We consider a real normed linear space. Let K be a convex set with as internal point such that K contains all its bounding points. Then K is the intersection of all the half spaces containing K that are determined by the supporting hyperplanes.

Lemma 3. In a normed linear space, let K and L be disjoint sets where K is compact and L is closed. Then $K - L$ is closed.

Theorem 5.17. If K and L are disjoint convex sets of a real normed linear space X, where K is compact and L is closed, then there is a hyperplane strictly separating K and L.

Note. The following result from $\mathbb{R}^n$ is super-geometric and easily visualized in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Theorem 5.18. Let K and L be two disjoint convex sets in $\mathbb{R}^n$. Then there is a hyperplane H separating K and L.

Note. Of course the proof of Theorem 5.18 is only valid in finite dimensions since it is based on induction involving $\dim(K)$ ($\ldots$ OK, "backwards induction"!).

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