6.2. Adjoints

Note. In this section, we extend the idea of the adjoint of an operator from the setting of Hilbert space (where we used the inner product to define the adjoint of an operator) to the more general setting of normed linear spaces where we make use of the dual space.

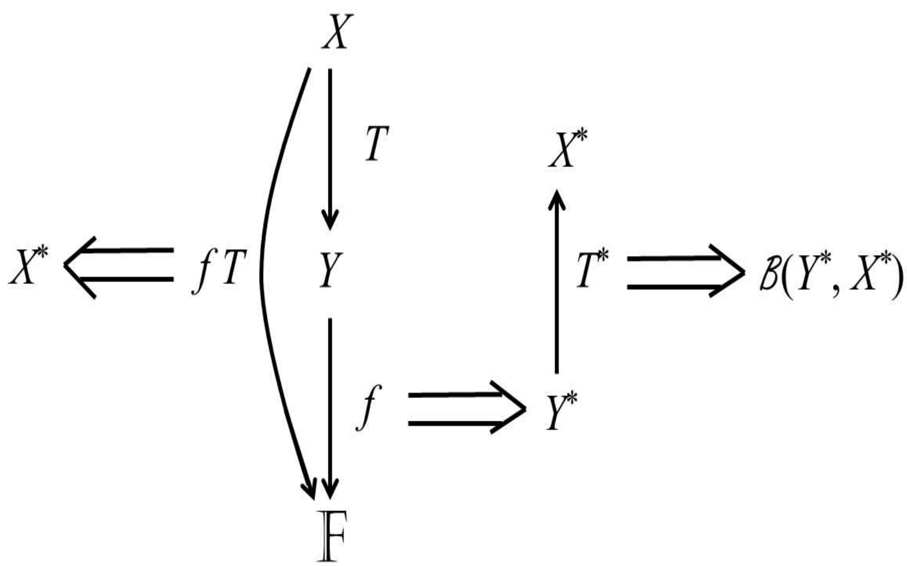
Definition. Given $T \in \mathcal{B}(X, Y)$ (i.e., T is a bounded linear operator from X to Y) in which X and Y are normed linear spaces, then a mapping T^* from Y^* to X^* for which $(T^*f)(x) = f(Tx)$ for all $f \in Y^*$ and for all $x \in X$ is the *adjoint* of T. That is, T^* maps $f \in Y^*$ to $fT \in X^*$.

Note. First, since $T \in \mathcal{B}(X, Y)$ is linear and $f \in Y^*$ is linear, then

$$\begin{aligned}
(T^*f)(\alpha x_1 + \beta x_2) &= f(T(\alpha x_1 + \beta x_2)) \text{ (definition of } T^*f) \\
&= f(\alpha T(x_1) + \beta T(x_2)) \text{ (since } T \text{ is linear)} \\
&= \alpha f(T(x_1)) + \beta f(T(x_2)) \text{ (since } f \text{ is linear)} \\
&= \alpha(T^*f)(x_1) + \beta(T^*f)(x_2) \text{ (definition of } T^*f).
\end{aligned}$$

So $T^*f : X \to \mathbb{F}$ is linear. In Theorem 6.6 below, we show that $\|T^*\| = \|T\|$, and so T^* is a bounded linear transformation. Therefore $T^* \in \mathcal{B}(Y^*, X^*)$.

Note. To clarify, we are dealing with four normed linear spaces: X, Y, X^*, and Y^*. We have the following mappings (represented as $\rightarrow$) and inclusions (we represent "an element of", $\in$, with $\Rightarrow$):



We have defined $T^* \in \mathcal{B}(Y^*, X^*)$ as mapping $f \in Y^*$ to $fT \in X^*$.

Note. The definition of adjoint here *does not* coincide with the definition from the Hilbert space setting in Section 4.5. There, we dealt with mappings T from H to H (instead of from X to Y). We then defined T^* using the inner product on H. On page 87 a mapping from H to H^* is defined (denoted J_H) which is conjugate linear instead of linear. We saw on page 90 that in finite dimensions, the matrix A representing T is related to matrix A^* representing T^* through the process of conjugate transpose. Example 6.5 shows that the current definition in finite dimensions implies that A and A^* are transposes of each other without the conjugation.

Note. A common way to deal with the difference between the normed linear space setting and the Hilbert space setting is to separately define a *Hilbert space adjoint* T^{H*} of T as $T^{H*} = C^{-1}T^*C$ where $C = \overline{J}_H$ (the conjugate of J_H). Then a Hilbert space adjoint is slightly different from a normed linear space adjoint (as we have), but directly related to the normed linear space setting. See Reed and Simon's *Functional Analysis*, page 186. (The conjugation comes with the fact that Reed and Simon define an inner product which is conjugate linear in the first position, instead of in the second position as we have.)

Example 6.5. Let T be a linear operator from X to Z where both X and Z are finite dimensional normed linear spaces. Let $\{b_1, b_2, \ldots, b_n\}$ be a basis for X and $\{c_1, c_2, \ldots, c_m\}$ be a basis for Z. Then T can be represented by an $m \times n$ matrix A_T (we know from Linear Algebra, or see Section 1.3). The jth column of A_T is $T(b_j)$. For $b_k \in \{b_1, b_2, \ldots, b_n\}$, define $J_X(b_k) \in X^*$ by defining $J_X(b_k)$ on $\{b_1, b_2, \ldots, b_n\}$ as

$$(J_X(b_k))(b_i) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

Then $\{J_X b_1, J_X b_2, \ldots, J_X b_n\}$ is a basis for X^*. Similarly, $\{J_X c_1, J_X c_2, \ldots, J_X c_m\}$ is a basis for Z^*. This is how a basis for a dual space (called a *dual basis*) is dealt with in the setting of normed linear spaces. We now find a matrix representation of T^* with respect to the bases of X^* and Z^*. We do so by applying $T^* : Z^* \to X^*$ to the basis elements of Z^*—this determines the columns of A_{T^*}. We then find that A_{T^*} is the transpose of A_T (without conjugates, as in the Hilbert space setting). This is shown in Exercise 6.17.

Note. The following result shows that adjoints in the normed linear space setting is similar to the behavior of adjoints in the Hilbert space setting. However, again, we have a slight difference concerning conjugation (see part (b)).

Theorem 6.6. Properties of the Adjoint in the Normed Linear Space Setting.
For all $S, T \in \mathcal{B}(X, Y)$, $A \in \mathcal{B}(Y, Z)$, and $\alpha \in \mathbb{F}$, we have

(a) $(S + T)^* = S^* + T^*$,

(b) $(\alpha T)^* = \alpha T^*$ (notice the absence of a conjugate of α),

(c) $(AT)^* = T^* A^*$, and

(d) $\|T^*\| = \|T\|$.

Note. The following result gives information about the range of T in terms of the nullspace of T^*.

Proposition 6.7. Let $T \in \mathcal{B}(X, Y)$ and $f \in Y^*$. Then $y \in \overline{R(T)}$ if and only if $f(y) = 0$ for all $y \in N(T^*)$.