6.3. Double Duals and Reflexivity

Note. In this section, we look at the dual of the dual of a space (called the double dual of the original space). A Banach space mapped in a certain way into its double deal, is called reflexive. We look at some properties of reflexive spaces.

Definition. Given a normed linear space $X$, the double dual of $X$, denoted $X^{**}$, is the dual of $X^*$.

Note. Let $x \in X$. Define an element $\hat{x} \in X^{**}$ as $\hat{x}(f) = f(x)$ for all $f \in X^*$. Notice that $f(x)$ and $\hat{x}(f)$ are elements of the scalar field $\mathbb{F}$. (So we start with $x \in X$ and then define $\hat{x}$ by letting the argument of $\hat{x}$ range over all $f \in X^*$.)

Theorem 6.8. The mapping $x \to \hat{x}$ (which maps $X$ to $X^{**}$) is a linear isometry.

Note. Theorem 6.8 shows that every normed linear space $X$ is a subspace of the space of operators $\mathcal{B}(X^*, \mathbb{F})$ (OK, there is a linear isometry between $X$ and a subspace of $\mathcal{B}(X^*, \mathbb{F})$). Recall that Theorem 2.15 says “If $Y$ is complete, then so is $\mathcal{B}(X, Y)$.” With $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then $X^* = \mathcal{B}(X, \mathbb{F})$ is complete, and so is $X^{**} = \mathcal{B}(X^*, \mathbb{F})$. This allows us to finish the proof of the “Completion Theorem” (Theorem 2.22) and show that for any normed linear space $X$, there is a completion—namely, $X^{**}$.
**Theorem 6.9. General Uniform Boundedness Principle.**

Let $A$ be a subset of a normed linear space $X$ such that for all $f \in X^*$ we have that $f(A)$ is a bounded set of scalars. Then $A$ is bounded.

**Definition.** A Banach space $X$ for which the embedding of $X$ in $X^{**}$ given by $x \to \hat{x}$ is surjective (onto) is reflexive.

**Note.** Since $X^{**}$ is complete, it would only make sense to discuss reflexive Banach spaces, and not reflexive normed linear spaces in general.

**Note.** The text claims (page 133) that properties of reflexive Banach spaces are similar to properties of Hilbert spaces (the text uses the term “behavior”). One example is Exercise 6.4: Let $S$ be a reflexive Banach space and $f$ a bounded linear functional on $X$. Then

$$
\|f\| = \sup\{|f(x)| \mid \|x\| \leq 1\} = \max\{|f(x)| \mid \|x\| \leq 1\}.
$$

That is, $f$ attains its maximum value on the unit disk.

**Note.** Most of the Banach spaces we encounter will be reflexive. An example of a nonreflexive space is $c_0$, the space of all sequences converging to 0. This is a separable space (one can show that the countable set of all sequences of rational numbers for which all but a finite number of entries are nonzero is dense in $c_0$). From Theorem 6.1, $c_0^* = \ell^1$ and $c_0^{**} = (\ell^1)^* = \ell^\infty$. Now $\ell^\infty$ is not separable (Proposition 2.42), so $c_0^{**} = \ell^\infty \neq c_0$ (well, there is not a surjective isometry).
Theorem 6.10. $L^p$ is reflexive for $1 < p < \infty$.

Theorem 6.11. A closed subspace of a reflexive Banach space is reflexive.

Theorem 6.12. A Banach space $X$ is reflexive if and only if $X^*$ is reflexive.