6.3. Double Duals and Reflexivity

Note. In this section, we look at the dual of the dual of a space (called the double dual of the original space). A Banach space mapped in a certain way into its double deal, is called reflexive. We look at some properties of reflexive spaces.

Definition. Given a normed linear space X, the *double dual* of X, denoted X^{**}, is the dual of X^*.

Note. Let $x \in X$. Define an element $\hat{x} \in X^{**}$ as $\hat{x}(f) = f(x)$ for all $f \in X^*$. Notice that $f(x)$ and $\hat{x}(f)$ are elements of the scalar field $\mathbb{F}$. (So we start with $x \in X$ and then define $\hat{x}$ by letting the argument of $\hat{x}$ range over all $f \in X^*$.)

Theorem 6.8. The mapping $x \to \hat{x}$ (which maps X to X^{**}) is a linear isometry.

Note. Theorem 6.8 shows that every normed linear space X is a subspace of the space of operators $\mathcal{B}(X^*, \mathbb{F})$ (OK, there is a linear isometry between X and a subspace of $\mathcal{B}(X^*, \mathbb{F})$). Recall that Theorem 2.15 says "If Y is complete, then so is $\mathcal{B}(X, Y)$." With $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, then $X^* = \mathcal{B}(X, \mathbb{F})$ is complete, and so is $X^{**} = \mathcal{B}(X^*, \mathbb{F})$. This allows us to finish the proof of the "Completion Theorem" (Theorem 2.22) and show that for any normed linear space X, there is a completion—namely, X^{**}.

Theorem 6.9. General Uniform Boundedness Principle.

Let A be a subset of a normed linear space X such that for all $f \in X^*$ we have that $f(A)$ is a bounded set of scalars. Then A is bounded.

Definition. A Banach space X for which the embedding of X in X^{**} given by $x \to \hat{x}$ is surjective (onto) is *reflexive*.

Note. Since X^{**} is complete, it would only make sense to discuss reflexive Banach spaces, and not reflexive normed linear spaces in general.

Note. The text claims (page 133) that properties of reflexive Banach spaces are similar to properties of Hilbert spaces (the text uses the term "behavior"). One example is Exercise 6.4: Let S be a reflexive Banach space and f a bounded linear functional on X. Then

$$\|f\| = \sup\{|f(x)| \mid \|x\| \leq 1\} = \max\{|f(x)| \mid \|x\| \leq 1\}.$$

That is, f attains its maximum value on the unit disk.

Note. Most of the Banach spaces we encounter will be reflexive. An example of a nonreflexive space is c_0, the space of all sequences converging to 0. This is a separable space (one can show that the countable set of all sequences of rational numbers for which all but a finite number of entries are nonzero is dense in c_0). From Theorem 6.1, $c_0^* = \ell^1$ and $c_0^{**} = (\ell^1)^* = \ell^\infty$. Now ℓ^∞ is not separable (Proposition 2.42), so $c_0^{**} = \ell^\infty \neq c_0$ (well, there is not a surjective isometry).

Theorem 6.10. L^p is reflexive for $1 < p < \infty$.

Theorem 6.11. A closed subspace of a reflexive Banach space is reflexive.

Theorem 6.12. A Banach space X is reflexive if and only if X^* is reflexive.

Revised: 5/20/2015