6.4. Weak and Weak* Convergence

Note. In this section, we define a new type of convergence of a sequence in a normed linear space X. The convergence depends heavily on the dual space X^*. Our exploration is shallow. A more detailed study (with heavy emphasis on L^p spaces) is given in Chapter 8 of Royden and Fitzpatrick's *Real Analysis* 4th Edition.

Definition. A sequence (x_n) in a normed linear space X *converges weakly* to $x \in X$ if the sequence of scalars $(f(x_n))$ converges to $f(x)$ for all $f \in X^*$.

Lemma. If (x_n) is convergent to x in X then (x_n) is weakly convergent to x.

Example 6.13. Of course, there are examples where a sequence converges weakly but does not converge. Consider ℓ^p where $1 < p < \infty$. Define $\delta_n \in \ell^p$ to be the sequence with a 1 in the nth position and 0's in all other positions. Then (δ_n) does not converge under the L^p norm since $\|\delta_n - \delta_m\| = 2^{1/p}$ for $n \neq m$ and so the sequence is not Cauchy. However, the sequence converges weakly to 0. Let $g \in (\ell^p)^* = \ell^q$ where $1/p + 1/q = 1$. Then

$$g(\delta_n) = \sum_{k=1}^{\infty} g(k)\delta_n = g(n)\delta_n = g(n)$$

(where we represent $g \in \ell^q$ as $g = (g(1), g(2), \ldots)$; we have also used the Riesz Representation Theorem for ℓ^p to represent $g(\delta_n)$ as the series given). Since $g \in \ell^q$, then $\|g\|_q = \left\{ \sum_{k=1}^{\infty} |g(k)|^q \right\}^{1/q}$ and so $\lim_{k\to\infty} |g(k)| = 0$. Therefore, $\lim_{n\to\infty} g(\delta_n) = \lim_{n\to\infty} g(n) = 0$ and so (δ_n) converges weakly to 0.

Proposition 6.14. Uniqueness of Weak Limits.

If (x_n) converges weakly to both x and y, then $x = y$.

Proof. Suppose (x_n) converges weakly to x and y. Assume $x \neq y$. Then by Corollary 5.6, there is some $f \in X^*$ such that $f(x) \neq f(y)$. But for this f, we need the sequence of scalars $f(x_n) \to x$ and $f(x_n) \to y$. However, limits of scalars are unique (recall, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$), a contradiction. So $x = y$. ∎

Proposition 6.15. Continuity of Operations.

For any sequence (x_n) which converges weakly to x, any sequence (y_n) which converges weakly to y, and any sequence of scalars (α_n) converging to α, we have:

(a) $(x_n + y_n)$ converges weakly to $x + y$,

(b) $(\alpha_n x_n)$ converges weakly to αx.

Note. The following result gives a relationship between weak convergence and regular ("strong") convergence in L^p spaces.

Theorem. The Radon-Riesz Theorem.

Let E be a measurable set and $1 < p < \infty$. Suppose $\{f_n\}$ converges weakly to f in $L^p(E)$. Then $\{f_n\}$ converges to f in $L^p(E)$ if and only if $\lim_{n\to\infty} \|f_n\| = \|f\|_p$.

Note. A proof of the Radon-Riesz Theorem can be found in Riesz and Sz.-Nagy's *Functional Analysis*, London: Blackie & Son Limited (1956) (reprinted by Dover Publishing in 1990). I have an online version of the proof at:

`http://faculty.etsu.edu/gardnerr/5210/notes/Radon-Riesz.pdf`

Note. We'll see that in ℓ^1, weak convergence is equivalent to convergence. To prove this, we need a new idea.

Definition. Let $x \in \ell^1$. Then x has a *hump* over the interval $[c, d]$ if

$$\sum_{k=1}^{d} |x(k)| \geq \frac{3}{5}\|x\|_1.$$

Note. The choice of 3/5 in the hump definition is somewhat arbitrary. Any value greater than 1/2 could be used to yield the same result we will get.

Proposition 6.16. If a sequence (x_n) in ℓ^1 converges weakly to x, then (x_n) converges to x with respect to the ℓ^1 norm. We take $\mathbb{F} = \mathbb{C}$.

Note. The following is a type of convergence for a sequence of functionals in X^*

Definition. Let X be a normed linear space. A sequence $(f_n) \subseteq X^*$ is *weak* convergent* to $f \in F^*$ if $(f_n(x))$ converges to $f(x)$ for all $x \in X$.

Note. We can show that the weak* limit of $(f_n) \subseteq X$ is unique.

Lemma. If $(f_n) \subseteq X^*$ is weak convergent to $f \in X^*$, then (f_n) is weak* convergent to f. That is, weak* convergence is weaker than weak convergence.

Proof. Suppose $(f_n) \subseteq X^*$ converges weakly to $f \in X^*$. Then for all $x \in X$,

$$\begin{aligned} f_n(x) &= \hat{x}(f_n) \text{ by definition of } \hat{x} \in X^{**} \\ &\to \hat{x}(f) \text{ since } f_n \to f \text{ weakly (replace } x_n \text{ with } f_n,\ x \text{ with } f, \text{ and } f \text{ with} \\ &\qquad\qquad x \text{ in the definition of weak convergence} \\ &= f(x) \text{ by definition of } \hat{x}. \end{aligned}$$

So (f_n) is weak* convergent to f. ∎

Note. IF space X is reflexive, then we can replace $\hat{x} \in X^*$ with $x \in X$ to show that weak* convergence implies weak convergence. Therefore weak and weak* convergence are equivalent on reflexive Banach spaces.

Note. The text uses weak* convergence as a segue into topological spaces, but we are skipping the topology chapter to explore the Spectral Theorem.

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