Chapter 8. The Spectrum

8.1. Introduction

Note. In this section we generalize the idea of an eigenvalue of a matrix (recall that every linear transformation in finite dimensions is represented by a matrix) to the "spectrum" of a bounded linear operator. In this chapter we explore properties of the spectrum and show how it gives information about its operator. Throughout this chapter we take $\mathbb{F} = \mathbb{C}$.

Note. In finite dimensions the spectrum and eigenvalues are the same. However, in infinite dimensions a bounded linear operator may not have eigenvalues, as the following shows.

Example 8.1. Let (x_n) be an orthonormal basis of ℓ^2. The *right shift operator* S is defined on the basis as $S(x_n) = x_{n-1}$, and so

$$S\left(\sum_{k=1}^{\infty} \alpha_k x_k\right) = \sum_{k=2}^{\infty} \alpha_{k-1} x_k = \sum_{k=1}^{\infty} \alpha_k x_{k+1}.$$

Suppose λ is an eigenvalue of S. Then $Sx = \lambda x$ for some nonzero $x = \sum_{k=1}^{\infty} \alpha_k x_k$. Let m be the smallest $m \in \mathbb{N}$ such that $\alpha_k \neq 0$. Then

$$Sx = \sum_{k=2}^{\infty} \alpha_{k-1} x_k = \sum_{k=1}^{\infty} \lambda \alpha_k x_k.$$

Then the coefficient of x_m is $\alpha_{m-1} = 0$ in the first series and $\lambda\alpha_m$ in the second series. So $\alpha_{m-1} = 0 = \lambda\alpha_m$ where $\alpha_m \neq 0$, and hence $\lambda = 0$. But then all coefficients of the x_n's are 0 and $x = 0$, a contradiction ($x = 0$ is not an eigenvector). So S has no eigenvalues.

Definition. For $T \in \mathcal{B}(X)$ (i.e., T is a bounded linear transformation from X to X), then T is *invertible* if there exists $T^{-1} \in \mathcal{B}(X)$ such that $TT^{-1} = T^{-1}T = I$ on X.

Note. We claim that when X is a Banach space and $T \in \mathcal{B}(X)$ is bijective (one to one and onto), then T^{-1} exists in $\mathcal{B}(X)$. However, if X is not a Banach space then it is possible for bijective $T \in \mathcal{B}(X)$ to not be invertible—of course the mapping T^{-1} exists, but T^{-1} may not be bounded.

Definition. The *spectrum* of any $T \in \mathcal{B}(X)$, denoted $\sigma(T)$, is the set of all scalars λ such that $T - \lambda I$ is <u>not</u> invertible in $\mathcal{B}(X)$.

Note. The spectrum consists of the following types of scalars:

(i) standard eigenvalues for which $T - \lambda I$ is not injective (one to one); then, of course, $T - \lambda I$ is not invertible,

(ii) scalars for which $T - \lambda I$ is injective (one to one) but not bounded below, and

(iii) scalars for which $T - \lambda I$ is bounded below (and then by Theorem 3.6 $R(T - \lambda I)$ is closed if X is complete) but $R(T - \lambda I)$ does not equal X (that is, $T - \lambda I$ is not surjective [onto]).

Definition. The values in the spectrum $\sigma(T)$ for $T \in \mathcal{B}(X)$ for which (i) above holds form the set of values called the *point spectrum* (so the point spectrum corresponds to the eigenvalues). The values for which (ii) holds <u>and</u> $R(T - \lambda I)$ is dense in X form the set of values called the *continuous spectrum*. The remaining values, for which (iii) or possibly (ii) hold (namely, we might have (ii) holding and $R(T - \lambda I)$ not dense in X) form the set of values called the *residual spectrum*. Define $\mathbb{C} \setminus \sigma(T)$ is the *resolvent set* of T.

Example 8.3. Let $f = (f(1), f(2), \ldots) \in \ell^2$. Define $M_f \in \mathcal{B}(\ell^2)$ as $M_f(g) = (f(1)g(1), f(2)g(2), \ldots)$. Then for any λ in the range of f, λ is an eigenvalue since $M_f(\delta_n) = f(n)\delta_n$ (so δ_n is the eigenvector corresponding to eigenvalue $\lambda = f(n)$). Any λ not in the range of f but in its closure is not an eigenvalue but is in the spectrum. This can be shown by taking $\epsilon > 0$ and such a λ. Then there is $\lambda_0 = f(k)$ for some $k \in \mathbb{N}$ such that $|\lambda - \lambda_0| < \epsilon$. Then $(M_f - \lambda I)\delta_k = (\lambda_0 - \lambda)\delta_k$ and this has norm less than ϵ. So $M_f - \lambda I$ is not bounded below (see the definition of "bounded below" on page 63), and so $M_f - \lambda I$ is not invertible by Theorem 3.6. So λ is in the spectrum of M_f.

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