8.2. Banach Algebras

Note. In this section we define an algebra on a linear space (not to be confused with an algebra *of sets*). As you should expect, a Banach algebra then involves completeness.

Definition. An *algebra* is a linear space A on which multiplication of (vector) elements of A is defined. For $x, y \in A$ we denote the *product* as xy. We require that for all $x, y, z \in A$ and scalars $\alpha \in \mathbb{F}$:

(i) Associative Law of Multiplication: $(xy)z = (z(yz)$.

(ii) Distributive Law of (Vector) Multiplication over (Vector) Addition:

$$x(y+z) = xy + zy \text{ and } (y+z)x = yz + zx.$$

(iii) Law Connecting (Vector) Multiplication and Scalar Multiplication: $(\alpha x)y = x(\alpha y) = \alpha(xy)$.

Definition. An element e in an algebra A is a *unit* if $ex = xe = x$ for all $x \in A$. An algebra with a unit is a *unital algebra*.

Note. If a unit exists, it is unique.

Definition. A *subalgebra* of an algebra A is a subspace of linear space A which is closed under multiplication. An element x of a unital algebra is *invertible* if there exists x^{-1} in the algebra such that $xx^{-1} = x^{-1}x = e$, where e is the unit of A.

Note. If x is invertible, then the inverse x^{-1} is unique.

Lemma. Products and Inverses Lemma.

Let x_1, x_2 be elements of a unital algebra.

(a) If x_1 has a left inverse and a right inverse, then x_1 is invertible.

(b) If x_1 and x_2 are invertible, then the product x_1x_2 is invertible.

(c) If the product x_1x_2 is invertible and $x_1x_2 = x_2x_1$, then each factor x_1 and x_2 is invertible.

Example. For normed linear space X, the space $\mathcal{B}(X)$ is an algebra with multiplication defined as function composition. Since the identity function is in $\mathcal{B}(X)$, then this is a unital algebra.

Note. The spectrum can be defined in a unital algebra A as

$$\sigma(x) = \{\lambda \in \mathbb{F} \mid x - \lambda e \text{ is not invertible in } Z\}.$$

Example 8.4. Let $A = C[a, b]$ (the continuous functions on $[a, b] \subset \mathbb{R}$). Define multiplication pointwise. Then $I(t) = 1$ for all $t \in [a, b]$ is the identity. Notice that $f \in C[a, b]$ is invertible if and only if $f(t) \neq 0$ for all $t \in [a, b]$. If λ is <u>not</u> in the range of f, then $(f - \lambda e)(t) = f(t) - \lambda \neq 0$ for all $t \in [a, b]$. So in this case, λ is <u>not</u> in the spectrum. Hence, for the multiplication operator M_f, the spectrum is the range of f: $\sigma(M_f) = R(f)$.

Definition. A *Banach algebra* A is an algebra on a complete normed linear space (i.e., a Banach space) such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathbb{F}$.

Note. Multiplication in a Banach algebra is continuous. That is, if $(x_n) \to x$ and $(y_n) \to y$, then $(x_n y_n) \to xy$.

Example. If X is a normed linear space, then $\mathcal{B}(X)$ is a Banach algebra. Completeness with respect to the sup norm of $\mathcal{B}(X)$ is given by Theorem 2.14. The inequality concerning the norm of a product is given by Proposition 2.8.

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