8.2. Banach Algebras

Note. In this section we define an algebra on a linear space (not to be confused with an algebra of sets). As you should expect, a Banach algebra then involves completeness.

Definition. An algebra is a linear space $A$ on which multiplication of (vector) elements of $A$ is defined. For $x, y \in A$ we denote the product as $xy$. We require that for all $x, y, z \in A$ and scalars $\alpha \in \mathbb{F}$:

(i) Associative Law of Multiplication: $(xy)z = (z(yz))$.

(ii) Distributive Law of (Vector) Multiplication over (Vector) Addition:

$$x(y + z) = xy + zy \quad \text{and} \quad (y + z)x = yz + zx.$$  

(iii) Law Connecting (Vector) Multiplication and Scalar Multiplication: $(\alpha x)y = x(\alpha y) = \alpha(xy)$.

Definition. An element $e$ in an algebra $A$ is a unit if $ex = xe = x$ for all $x \in A$. An algebra with a unit is a unital algebra.

Note. If a unit exists, it is unique.
**Definition.** A subalgebra of an algebra $A$ is a subspace of linear space $A$ which is closed under multiplication. An element $x$ of a unital algebra is invertible if there exists $x^{-1}$ in the algebra such that $xx^{-1} = x^{-1}x = e$, where $e$ is the unit of $A$.

**Note.** If $x$ is invertible, then the inverse $x^{-1}$ is unique.

**Lemma. Products and Inverses Lemma.**

Let $x_1, x_2$ be elements of a unital algebra.

(a) If $x_1$ has a left inverse and a right inverse, then $x_1$ is invertible.

(b) If $x_1$ and $x_2$ are invertible, then the product $x_1x_2$ is invertible.

(c) If the product $x_1x_2$ is invertible and $x_1x_2 = x_2x_1$, then each factor $x_1$ and $x_2$ is invertible.

**Example.** For normed linear space $X$, the space $\mathcal{B}(X)$ is an algebra with multiplication defined as function composition. Since the identity function is in $\mathcal{B}(X)$, then this is a unital algebra.

**Note.** The spectrum can be defined in a unital algebra $A$ as

$$\sigma(x) = \{\lambda \in \mathbb{F} \mid x - \lambda e \text{ is not invertible in } Z\}.$$
Example 8.4. Let $A = C[a,b]$ (the continuous functions on $[a,b] \subset \mathbb{R}$). Define multiplication pointwise. Then $I(t) = 1$ for all $t \in [a,b]$ is the identity. Notice that $f \in C[a,b]$ is invertible if and only if $f(t) \neq 0$ for all $t \in [a,b]$. If $\lambda$ is not in the range of $f$, then $(f - \lambda e)(t) = f(t) - \lambda \neq 0$ for all $t \in [a,b]$. So in this case, $\lambda$ is not in the spectrum. Hence, for the multiplication operator $M_f$, the spectrum is the range of $f$: $\sigma(M_f) = R(f)$.

Definition. A Banach algebra $A$ is an algebra on a complete normed linear space (i.e., a Banach space) such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in F$.

Note. Multiplication in a Banach algebra is continuous. That is, if $(x_n) \rightarrow x$ and $(y_n) \rightarrow y$, then $(x_ny_n) \rightarrow xy$.

Example. If $X$ is a normed linear space, then $B(X)$ is a Banach algebra. Completeness with respect to the sup norm of $B(X)$ is given by Theorem 2.14. The inequality concerning the norm of a product is given by Proposition 2.8.