8.3. General Properties of the Spectrum

Note. In this section we elaborate on the spectrum by stating the Spectral Mapping Theorem and define the spectral radius. The results of the section concern Banach algebras.

Theorem 8.5. The Spectral Mapping Theorem.

Let p be a polynomial. Let X be a linear space. Then $\mu \in \sigma(p(x))$ if and only if $\mu = p(\lambda)$ for some $\lambda \in \sigma(x)$, where $x \in X$.

Example 8.6. Applications of the Spectral Mapping Theorem.

(a) Let $x \in X$, X an algebra, such that $x^n = 0$ for some $n \in \mathbb{N}$. Such x is *nilpotent*. Consider $p(t) = t^n$ (for the n above). Since $p(x) = x^n = 0$ and the spectrum of the 0 operator is $\{0\}$ ($0 - \lambda e$ has inverse $(-1/\lambda)e$, unless $\lambda = 0$), then by the Spectral Mapping Theorem we have that $\mu \in \sigma(p(x)) = \sigma(0) = \{0\}$ if and only if $0 = \mu \in p(\lambda)$ for some $\lambda \in \sigma(x)$. So $p(\lambda) = \lambda^n = 0$ and hence $\lambda = 0$. So for x nilpotent, $x - \lambda e$ is invertible unless $\lambda = 0$.

(b) Let $x \in X$, X an algebra, such that $x^2 = x$. Such x is *idempotent*. Consider $p(t) = t^2 - t$. By the Spectral Mapping Theorem, $\mu \in \sigma(p(x)) = \sigma(0) = \{0\}$ (for idempotent x) if and only if $0 = \mu \in p(\lambda)$ for some $\lambda \in \sigma(x)$. Since the zeros of p are 0 and 1, then the values of $\lambda \in \sigma(x)$ must be 0 and 1. That is, $\sigma(x) \subseteq \{0, 1\}$.

Proposition 8.7. Suppose x is invertible. Then $\lambda \in \sigma(x)$ if and only if $\lambda^{-1} \in \sigma(x^{-1})$.

Note. In any unitary algebra, e is invertible (it is its own inverse), so we would expect that elements of the algebra close to e should be invertible. This is quantified in the following.

Proposition 8.8. Let X be a (complete) Banach algebra. if $\|e - x\| < 1$, then x is invertible.

Proposition 8.9. The set of invertible elements of a Banach algebra is an open set.

Theorem 8.10. Let X be a Banach algebra. Then for all $x \in X$, $\sigma(x)$ is a compact subset of $\mathbb{C}$.

Definition. The *spectral radius* of an element x of a Banach algebra, denoted $r(x)$, is

$$r(x) = \sup\{|\lambda| \mid \lambda \in \sigma(x)\}.$$

If $\sigma(x) = \varnothing$, we take $r(x) = 0$.

Proposition 8.11. Let X be a Banach algebra. Then for any $x \in X$, the spectral radius of x satisfies

$$r(x) \leq \inf\{\|x^n\|^{1/n} \mid n \in \mathbb{N}\}.$$

Note. Our next task is to actually compute $(x - \lambda e)^{-1}$ for $\lambda \notin \sigma(x)$.

Definition. A sequence of positive real numbers (a_n) is *submultiplicative* if $a_{n+m} \leq a_n a_m$ for all $n, m \in \mathbb{N}$.

Theorem 8.12. If (a_n) is a submultiplicative sequence of positive real numbers, then $(a_n^{1/n})$ converges to $\inf\{a_n^{1/n} \mid n \in \mathbb{N}\}$.

Note. We use Theorem 8.12 to represent $(x - \lambda e)^{-1}$ as a series in the following result.

Theorem 8.13. If $\inf\{\|a^n\|^{1/n} \mid n \in \mathbb{N}\} < |\lambda|$ then $(x - \lambda e)$ is invertible and

$$(x - \lambda e)^{-1} = -\sum_{k=0}^{\infty} \frac{x^k}{\lambda^{k+1}}.$$

Definition. An element x in a Banach algebra that satisfies $\lim_{n\to\infty} \|x^n\|^{1/n} = 0$ is *quasi-nilpotent*.

Note. We now present two "deeper properties" of the spectral radius and the spectrum. First, we need a fundamental result from complex analysis.

Theorem 8.14. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a complex-valued function of a complex variable such that ϕ is differentiable at all points of an open disc $\{z \in \mathbb{C} \mid |z| < r\}$. Then there is a unique sequence of complex numbers $(a_n)_{n=1}^{\infty}$ such that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges to $\phi(z)$ at all points of this disc.

Note. The following result shows that the bound on the spectral radius given in Proposition 8.11 in fact reduces to an equality.

Theorem 8.15. For all elements x in a Banach algebra A, $r(x) = \inf\{\|x^n\|^{1/n} \mid n \in \mathbb{N}\}$.

Theorem 8.16. For all elements x of a Banach algebra, $\sigma(x) \neq 0$.

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