8.4. Numerical Range

Note. We now introduce a new set of complex numbers which, for a bounded linear operator on a Hilbert space, contains the spectrum of the operator.

Definition. Let H be a Hilbert space. For $T \in \mathcal{B}(H)$, define the *numerical range* of T as the set $\{\langle Tx, x\rangle \mid x \in H, \|x\| = 1\}$. The *numerical radius* of T, denoted $w(T)$, is the supremum of the absolute values of the elements in the numerical range: $w(T) = \sup\{|\langle Tx, x\rangle| \mid x \in H, \|x\| = 1\}$.

Note. By the Cauchy-Schwartz Inequality (Theorem 4.3), for all $x \in H$ with $\|x\| = 1$, we have

$$|\langle Tx, x\rangle| \leq \|Tx\|\|x\| = \|Tx\| \leq \|T\|\|x\| = \|T\|.$$

So $w(T) \leq \|T\|$.

Proposition 8.17. For H a Hilbert space and $T \in \mathcal{B}(T)$, then $\sigma(T)$ is contained in the closure of the numerical range of T.

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