8.4. Numerical Range

**Note.** We now introduce a new set of complex numbers which, for a bounded linear operator on a Hilbert space, contains the spectrum of the operator.

**Definition.** Let $H$ be a Hilbert space. For $T \in \mathcal{B}(H)$, define the *numerical range* of $T$ as the set $\{\langle Tx, x \rangle \mid x \in H, \|x\| = 1\}$. The *numerical radius* of $T$, denoted $w(T)$, is the supremum of the absolute values of the elements in the numerical range: $w(T) = \sup\{|\langle Tx, x \rangle| \mid x \in H, \|x\| = 1\}$.

**Note.** By the Cauchy-Schwartz Inequality (Theorem 4.3), for all $x \in H$ with $\|x\| = 1$, we have

$$|\langle Tx, x \rangle| \leq \|Tx\|\|x\| = \|Tx\| \leq \|T\|\|x\| = \|T\|.$$

So $w(T) \leq \|T\|$.

**Proposition 8.17.** For $H$ a Hilbert space and $T \in \mathcal{B}(T)$, then $\sigma(T)$ is contained in the closure of the numerical range of $T$.  

Revised: 5/12/2017