8.5. Spectrum of a Normal Operator

Note. In this section, we discuss the spectrum and numerical radius of several of the special operators introduced in Section 4.6 (namely, self-adjoint, normal, and unitary).

Recall. The *adjoint* of operator $T \in \mathcal{B}(H)$ (where H is a Hilbert space) is the unique operator T^* (see Theorem 4.24) such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$. Operator T is *self-adjoint* if $T = T^*$, *normal* if $TT^* = T^*T$, is *unitary* if $TT^* = T^*T = I$, and is a *projection* if $T = T^*$ and $T^2 = T$. T is *positive* if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Proposition 8.18. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ where the spectrum of H is $\sigma(T)$.

(a) If T is self-adjoint then $\sigma(T) \subseteq \mathbb{R}$.

(b) If T is a positive operator then $\sigma(T)$ consists of nonnegative real numbers.

(c) If T is a projection then $\sigma(T) \subseteq \{0, 1\}$.

(d) If T is a unitary operator, then $\sigma(T) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}$.

Note. We now turn to self-adjoint operators and consider the numerical radius and specific elements of $\sigma(T)$.

Lemma 8.19. If T is a self-adjoint operator on a Hilbert space H, then for all unit vectors x and y in H, we have $\text{Re}(\langle Tx, y \rangle) \leq w(T)$.

Proposition 8.20. If T is a self-adjoint operator on a Hilbert space, then $w(T) = \|T\|$.

Proposition 8.21. For a self-adjoint operator T on a Hilbert space, either $\|T\| \in \sigma(T)$ or $-\|T\| \in \sigma(T)$.

Note. Propositions 8.20 and 8.21 combine to show that for self-adjoint T, the spectral radius is $r(T) = \|T\|$. In fact, the same result holds for normal operators, as shown in the second of the following two results.

Proposition 8.22. If T is a normal operator on a Hilbert space, then $\|T^n\| = \|T\|^n$.

Theorem 8.23. If T is a self adjoint or normal operator on a Hilbert space, then $r(T) = \|T\|$.

Note. We now consider eigenvalues and eigenspaces for a self-adjoint operator.

Proposition 8.24. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be self-adjoint. Then eigenspaces corresponding to distinct eigenvalues of T are orthogonal.

Proposition 8.25. Let H be a Hilbert space and $T \in \mathcal{B}(H)$ be self-adjoint. If H is separable, then the number of distinct eigenvalues of T is either finite or countably infinite.

Note. In fact, Propositions 8.24 and 8.25 also hold for normal operators, as is to be shown in Exercise 8.11.

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