8.6. Functions of Operators

Note. In this section, we consider properties of polynomials evaluated at an operator and a continuous function evaluated at an operator, where the operator is a bounded self adjoint operator on a Hilbert space.

Note. For T an operator from a linear space to itself and for a polynomial $p(t) = \sum_{k=1}^{\infty} a_t t^k$ we can define the operator $p(T) = a_0 I a_1 T + a_2 T^2 + \cdots + a_n T^n$. In fact, for T a self adjoint bounded operator on a Hilbert space and f any continuous function on the spectrum $\sigma(T)$, then the following result let's us define an operator $f(T)$. This is from the area of "continuous functional calculus." This is covered in more depth in Reed and Simon's *Functional Analysis*, Section VII.1 (The Continuous Functional Calculus).

Proposition 8.26. Let T be a bounded, self adjoint operator on a Hilbert space H. Then the mapping that sends polynomials $p \in C(\sigma(T))$ (continuous functions on the spectrum of T) with the sup norm into $p(T) \in \mathcal{B}(H)$, sending $f \in C(\sigma(T))$ to $f(T)$ such that the following properties hold. For all $f, g \in C(\sigma(T))$ and any $\alpha \in \mathbb{C}$,

(i) $(f + g)(T) = f(T) + g(T)$ and $(\alpha f)(T) = \alpha f(T)$,

(ii) $\overline{f}(T) = (f(T))^*$ where $\overline{f}$ is the conjugate of function f,

(iii) $(fg)(T) = f(T)g(T)$, and

(iv) for any $S \in \mathcal{B}(H)$ such that $ST = TS$ we have $Sf(T) = f(T)S$.

Note. By Theorem 8.26(ii), if f is a ral valued function then $f(T)$ is self adjoint (recall that T is self adjoint by hypothesis).

Note. Theorem 8.26 also holds for normal operators. Promislow references Kadison and Rinrose's *Fundamentals of the Theory of Operator Algebras* (1983) for details.

Example 8.27. Suppose Hilbert space H is n-dimensional and self adjoint operator T has matrix representation (with respect to the standard basis, say),

$$A_T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

If p is a polynomial then

$$p(T) = p(A_T) = \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & p(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(\lambda_n) \end{bmatrix}$$

and for f a limit of a sequence of polynomials,

$$f(T) = \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_n) \end{bmatrix}.$$

A similar argument can be given for any T such that A_T is diagonolizable (this is one of the reasons to diagonalize a matrix).

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