8.7. Brief Introduction to C^*-Algebras

Note. As commented on page 170, for X a Banach space, $\mathcal{B}(X)$ is a Banach algebra where multiplication is defined as operator composition. In this section, we define a special kind of Banach algebra which is a useful tool in the study of Hilbert space operators.

Definition. Let H be a Hilbert space and set $\mathcal{A}$ a closed subalgebra of $\mathcal{B}(H)$. Suppose $\mathcal{A}$ is closed under adjoints (i.e., $T \in \mathcal{A}$ implies $T^* \in \mathcal{A}$). Then $\mathcal{A}$ is a C^*-algebra.

Note. The primary question of interest here involves determining which Banach algebras are isomorphic to C^*-algebras. Such a Banach algebra must have a mapping to itself which sends each x to its adjoint X^* and preserves addition, multiplication, and scalar multiplication—that is, the mapping must satisfy the properties of Theorem 4.26. In fact, any Banach algebra which satisfies (a), (b), (c), (d), and (f) of Theorem 4.26 is isometric to a C^*-algebra (property (e) follow from H).

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