Chapter 9. Compact Operators

9.1. Introduction and Basic Definitions

Note. In this section, we define a class of operators, called "compact operators." The behavior of compact operators are similar to the behavior of operators in finite dimensions. In fact, all operators on finite dimensional spaces are compact (thanks to the Heine-Borel Theorem).

Definition. A subset K of a normed linear space is *relatively compact* if $\overline{K}$ is compact.

Note. Reed and Simon in *Functional Analysis* (1980) use the term "precompact" instead of "relatively compact."

Lemma 9.1.A. A subset K of a normed linear space is relatively compact if and only if any sequence in K has a convergent subsequence (where the limit if an element of the normed linear space, but not necessarily in K).

Definition. Let X and Y be normed linear spaces. An operator $T \in \mathcal{B}(X, Y)$ is a *compact operator* if, for all bounded sets $B \subseteq X$, the set $T(B)$ is relatively compact in Y.

Lemma 9.1.B. Let X and Y be normed linear spaces. Then $T \in \mathcal{B}(X, Y)$ is a compact operator if an only if given any bounded sequence (x_n) in X, the sequence (Tx_n) has a convergent subsequence.

Lemma 9.1.C. Let X and Y be normed linear spaces. Then $T \in \mathcal{B}(X, Y)$ is a compact operator if and only if $T(B(1))$ is relatively compact in Y, where $B(1) = \{x \in X \mid \|x\| < 1\}$.

Note. The proof of Lemma 9.1.C is to be given in Exercise 9.1.

Note. A relatively compact set is bounded (recall that all compact sets are closed and bounded—see The Compact Set Theorem in Section 2.2 of the class notes), so all compact operators are bounded. In finite dimensions, all bounded sets are relatively compact (Heine-Borel Theorem), so any bounded operator in finite dimensions is a compact operator (and conversely).

Revised: 5/16/2017