9.2. Compactness Criteria in Metric Spaces

Note. In this section, we introduce the ideas of relative compactness and total boundedness. We relate these ideas in metric spaces, Banach spaces, and function spaces. We define "equicontinuity" and prove the Arzela-Ascoli Theorem.

Definition. If A and B are subsets of a metric space (X, d) and if $\varepsilon \geq 0$, then we say "$A \subseteq^\varepsilon B$" if, given any $a \in A$, there is an element $b \in B$ with $d(a, b) \leq \varepsilon$.

Note. Promislow calls this " approximately contained in" and observes that

$$A \subseteq^\varepsilon B \text{ and } B \subseteq^\delta C \text{ implies that } A \subseteq^{\varepsilon+\delta} C.$$

Definition. A set A in a metric space is *totally bounded* if for any $\varepsilon > 0$ there is a finite set F such that $A \subseteq^\varepsilon F$. The set F is a *ε-net* for A.

Note. A totally bounded set A is bounded. This is because for F a ε-net of A, we have $\mathrm{diam}(A) \leq \mathrm{diam(F)} + 2\varepsilon$ and $\mathrm{diam}(F) < \infty$ since F is finite. However, in ℓ^2, the set $\{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots) \ldots\}$ is bounded but not totally bounded.

Proposition 9.1. Set A in a metric space is totally bounded if and only if any sequence (a_n) of points in A has a Cauchy subsequence.

Corollary 9.2.A. If set K in a normed linear space is relatively compact then K is totally bounded. In a complete space any totally bounded set is relatively compact.

Proof. If set K in a normed linear space is relatively compact then, by definition, any sequence in K has a convergent subsequence. Then K is totally bounded by Proposition 9.1, since convergent sequences are Cauchy.

In a complete space (where Cauchy sequences converge) any totally bounded set is relatively compact (again, by Proposition 9.1). ∎

Note. Recall that a compact set in a normed linear space is closed and bounded by The Compact Set Theorem (see the class notes for Section 2.2). The next result classifies relatively compact sets in a Banach space. Notice that a relatively compact set is totally bounded, as just noted, and totally bounded sets are bounded.

Proposition 9.2. A bounded set A of a Banach space X is relatively compact if and only if for any $\varepsilon > 0$ there is a finite dimensional subspace Y of X with $A \subseteq^{\varepsilon} Y$.

Note. We now apply Proposition 9.2 to function spaces.

Proposition 9.3. Let S be a set and $B(S)$ the set of functions from S to field $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) under the sup norm. Suppose that A is a bounded subset of $B(E)$ satisfying the following: For any $\varepsilon > 0$, we can partition S into a finite number of pairwise disjoint subsets $S_1, S_2, \ldots, S_n$ such that, given any i, any two points $s, t \in S_i$, and any $f \in A$, we have $|f(s) - f(t)| \leq \varepsilon$. Then A is relatively compact (in $B(S)$).

Note. Next, we will use Proposition 9.3 to address relative compactness in the space of continuous functions on S where S is a compact metric space (in the Arzela-Ascoli Theorem). To do so, we need a definition.

Definition. A set A of functions defined on a metric space S is *equicontinuous at a point* $t_0 \in S$ if for any given $\varepsilon > 0$ there is $\delta > 0$ such that $d(t, t_0) < \delta$ implies that $|f(t) - f(t_0)| < \varepsilon$ for all $f \in A$. Set A is an *equicontinuous set* if it is equicontinuous at all points of S.

Note. Equicontinuity of a *set of functions* is similar to uniform continuity *on a set*. However, in uniform continuity there is a single function and the input values range of a set of "points"; in equicontinuity at a point there is a single given input value ("point") but the functions vary over a set of functions (each considered at the given point).

Example 9.4. Let S be any subset of a normed linear space X and let A be a bounded subset of X^*, say each linear functional in X^* is bounded by K (so $T \in A$ implies $\|T\| \leq K$). Then A is equicontinuous on S since any $s \in S$ and $f \in A$ we can we can for given $\varepsilon > 0$ choose $\delta - \varepsilon/K$.

Theorem 9.5. **Arzela-Ascoli Theorem.**
If S is a compact metric space, a subset A of $C(S)$ (the set of continuous real valued or complex valued functionals on S) is relatively compact if and only if it is bounded and equicontinuous.

Note. We now give a condition for the relative compactness of a set in ℓ^p.

Theorem 9.6. Let A be a bounded subset of ℓ^p that has uniformly small tails. That is, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $f \in A$, $\sum_{i=N}^{\infty} |f(i)|^p < \varepsilon$. Then A is relatively compact.

Note. Recall that operator $T \in \mathcal{B}(X, Y)$ (where X and Y are normed linear spaces) is compact if for all bounded sets in X, $T(B)$ is relatively compact. To show that T is compact, it is sufficient to show that $T(B(1))$ is relatively compact (where $B(1)$ is the open unit ball); see the comment on page 187. We conclude this section by giving two examples of compact operators.

Theorem 9.7. The multiplication operator M_f on ℓ^p is compact if and only if $f(n) \to 0$.

Example 9.8. Let k be a continuous real valued function on the closed unit square of $\mathbb{R}^2$. Define K on $C([01,])$ by $K(f(s)) = \int_0^1 k(s,t)f(t)\,dt$. Since $k(s,t)$ is bounded (a continuous function on a compact set) then operator K is bounded: Let $\max|k(s,t)| = M$, then $|K(f(s))| = \left|\int_0^t k(s,t)f(t)\,dt\right| \le \int_0^1 |k(s,t)||f(t)|\,dt \le M\int_0^1 |f(t)|\,dt \le M \max|f(t)|$, so $\|K\| \le M$ (here we use the sup norm).

Theorem 9.9. The operator K given in Example 9.8 if compact.

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