9.3. New Compact Operators from Old

Note. In this section, we show that a scalar multiple, sum, and composition of compact operators are compact operators. Also, a limit of compact operators is compact and the adjoint of a compact operator is compact.

Proposition 9.10. Compactness and Algebraic Properties.
Suppose we are given normed linear spaces X, Y, and Z, a scalar α, operators $S, T \in \mathcal{B}(X, Y)$, and $A \in \mathcal{B}(Y, Z)$. We have:

(a) If T is compact, then so is αT.

(b) If S and T are compact, then so is $S + T$.

(c) If T is compact, then so is $A \circ T = AT$.

(d) If A is compact, then so is $A \circ T = AT$.

Note. Parts (a) and (b) of Proposition 9.10 show that the compact operators in $\mathcal{B}(X, Y)$ themselves form a linear space.

Proposition 9.11. Compactness and Limits.
If $T = \lim T_n$, in which (T_n) is a sequence of compact operators in $\mathcal{B}(X, Y)$, then T is compact.

Proposition 9.12. Compactness and Adjoints.
Let $T \in \mathcal{B}(X,Y)$. If T is compact, then its adjoint T^* is compact. If Y is complete, then compactness of T^* implies that of T.

Note. In Exercise 9.8 it is to be shown that on a Hilbert space any compact operator is the limit of a sequence of finite-rank operators (that is, operations with finite dimensional range). This same result holds for a Banach space with a Schauder basis (see J. R. Giles' *Introduction to the Analysis of Normed Linear Spaces*, Cambridge University Press (2000)). "A long outstanding question" (Promislow, page 193) was whether the result holds for any Banach space. Per Enflo in "A Counterexample to the Approximation Problem in Banach Spaces," *Acta Math.* **130**, 309–317 (1973) answered the question in the negative by giving a compact operator that is not the limit of finite rank operators. Enflo's counterexample involved the separable Banach space that does not have a Schauder basis (which we mentioned in Section 2.11).

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