9.4. Spectrum of a Compact Operator

Note. In this section we consider the spectrum of a compact operator $T \in \mathcal{B}(X)$ where X is a Banach space. The "big result" is Theorem 9.16 where the spectrum is described, the eigenvalues are enumerated, and the eigenspaces are all shown to be finite dimensional.

Note. If Banach space X is infinite dimensional, then a compact operator $T \in \mathcal{B}(X)$ cannot be invertible. If it were, then $T^{-1}T = i$ would be a compact operator by Theorem 9.10(c). But $I(B(1)) = B(1)$ and in an infinite dimensional space, $\overline{B(1)}$ is not compact by Theorem 2.34 (Riesz's Theorem), so $B(1)$ is not relatively compact and I is *not* compact, contradicting the supposition that T is invertible. That is, $T - 0I$ is not invertible and so $\lambda = 0$ is in the spectrum of T, $\sigma(T)$. In Theorem 9.16 we'll see that the nonzero elements of $\sigma(T)$ are all eigenvalues and each corresponding eigenspace is finite dimensional.

Lemma 9.13. Let S be a linear operator from any linear space to itself. Consider the nested subsequences of subspace: $R(S) \supset R(S^2) \supset R(S^3) \supset \cdots$ and $N(S) \subset N(S^2) \subset N(S^3) \subset \cdots$.

(a) Suppose that S is one to one (injective) but not onto (surjective). Then all inclusions in the range sequence are strict.

(b) Suppose that S is onto (surjective) but not one to one (injective). Then all inclusions in the null space sequence are strict.

Lemma 9.14. Let X be a Banach space and let $Y \in \mathcal{B}(X)$ be compact. Let (λ_n) be a sequence of real or complex scalars. Suppose we have a strictly increasing sequence of (topologically) closed subspaces of X: $Y_1 \subset Y_2 \subset Y_3 \subset \cdots$ such that $(T - \lambda_n I)Y_n \subset Y_{n-1}$ for all $n > 1$. Then $(|\lambda_n|)$ is not bounded below (in the sense given in Section 3.4). The same conclusion holds if we have a strictly decreasing sequence of closed subspaces of X: $Z_1 \supset Z_2 \supset Z_3 \supset \cdots$ such that $(T - \lambda_n I)Z_n \supset Z_{n+1}$ for all $n \in \mathbb{N}$.

Corollary 9.4.A. Let λ be a nonzero scalar and let $T \in \mathcal{B}(X)$ be compact where X is a Banach space. If $S = T - \lambda I$ is onto then it is one to one.

Note. We want a result similar to Corollary 9.4.A which implies that $S = T - \lambda I$ is onto if it is one to one. Additional properties are needed involving the range of $T - \lambda I$; we need $R(T - \lambda I)$ to be closed in order to apply Lemma 9.14.

Proposition 9.15. Let X be a Banach space and let $T \in \mathcal{B}(X)$ be a compact operator. Then the range $R(T - I)$ is (topologically) closed. (This is the range of $R - \lambda I$ where $\lambda = 1$.)

Note. Proposition 9.15 also holds if we replace I with λI, so that $R(T - \lambda I)$ is closed for compact T and scalar λ. We simply replace $S = T - I$ in the proof with $R = T - \lambda I$. The subsequence (x_{n_k}) is replaced with the subsequence (x_{n_k}) and we get $\lambda x_{n_k} \to 0$ so that $x_{n_k} \to 0$ still. The remainder of the proof carries over.

Corollary 9.4.B. Let λ be a nonzero scalar and let $T \in \mathcal{B}(X)$ be compact where X is a Banach space. If $S = T - \lambda I$ is one to one then it is onto.

Note. Corollaries 9.4.A and 9.4.B combine to give:

> Let λ be a nonzero sclaar and let $T \in \mathcal{B}(X)$ be compact where X is a Banach space. Then $S = T - \lambda I$ is ont to one if and only if it is onto.

Theorem 9.16. Let T be a compact operator in $\mathcal{B}(X)$, in which X is a Banach space. Then, the nonzero elements of the spectrum of T are eigenvalues. There are only countably many eigenvalues, and, in the case of infinitely many, they form a sequence tending to 0. The eigenspaces are all finite-dimensional.

Note. As mentioned in the second note in the class notes for this section, if X is an infinite dimensional Banach space and T is a compact operator in $\mathcal{B}(X)$, then 0 is in the spectrum of T. In this setting, if $T = T - \lambda I$ is not one to one then 0 is an eigenvalue and so is in the point spectrum. If $T = T - \lambda I$ is one to one then, since T is not invertible, by Theorem 3.6 T is not bounded below. Then 0 is in the continuous spectrum if $R(T) = R(T - \lambda I)$ is dense in X; 0 is in the residual spectrum if $R(T) = R(T - \lambda I)$ is not dense in X.

Revised: 5/16/2017