9.5. Compact Self Adjoint Operators on Hilbert Spaces

Note. In this section we give a spectral theorem for a compact self adjoint operator on a Hilbert space.

Note. Of course $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces. We know that every linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ (or $\mathbb{C}^n$ to $\mathbb{C}^n$) is represented by an $n \times n$ matrix, A_T (see my class notes for Linear Algebra [MATH 2010], Section 2.3, the "Standard Matrix Representation of Linear Transformations": `http://faculty.etsu.edu/gardnerr/2010/c2s3.pdf`). A real symmetric matrix is (real) diagonalizable if it is symmetric (see the "Fundamental Theorem of Real Symmetric Matrices": `http://faculty.etsu.edu/gardnerr/2010/c6s3.pdf`). A complex matrix is diagonalizable if it is conjugate symmetric (that is, $a_{ij} = \overline{a}_{ji}$). As observed in Section 4.6 (see the class notes for this section, page 3) these matrices correspond to self adjoint operators on finite dimensional spaces. We now turn to infinite dimensional spaces.

Definition. A subspace M of a linear space is *invariant* under linear operator T if $TM \subset M$.

Proposition 9.17. If M is invariant for compact, self adjoint operator T on a Hilbert space then $M^\perp$ is invariant for T. Moreover, the restrictions of T to both M and $M^\perp$ are also self adjoint.

Theorem 9.18. Spectral Theorem for Compact, Self Adjoint Operators. Let T be a compact, self adjoint operator on a Hilbert space H. There is a sequence (either finite or countably infinite) of mutually orthogonal closed subspaces (M_n) whose closed linear span is all of H. There is a corresponding sequence (λ_n) of real numbers which if countably infinite converges to 0. For all n and $x \in M_n$, we have $Tx = \lambda_n x$. Moreover, if $\lambda_n \neq 0$ then M_n is finite dimensional.

Note. Theorem 9.18 also holds for normal operators (though the eigenvalues may not be real). This is to be proved in Exercise 9.11.

Theorem 9.19. For T a compact, self adjoint operator on Hilbert space H, $T = \sum_n \lambda_n E_{\lambda_n}$ in which E_{λ_n} is the projection onto M_n where M_n is the eigenspace associated with λ_n.

Note. Suppose H is an infinite dimensional separable Hilbert space and let T be a compact, self adjoint operator on H. Then there is a sequence (λ_n) of eigenvalues of T and eigenspaces M_n such that H is the closed linear span of the M_n's by Theorem 9.18. Let B_n be an orthonormal basis for eigenspace M_n (so each B_n is finite, unless 0 is an eigenvalue in which case the eigenspace $N(T)$ need not be finite dimensional according to Theorem 9.18). Take the union of all these bases, say $B = \{e_k \mid k \in \mathbb{N}\} = \cup_n B_n$. Let μ_k be the eigenvalue corresponding to the eigenvector e_k (this yields sequence $(\mu_k)_{k=1}^{\infty}$ with μ_k's repeated according to the dimension of the corresponding eigenspace). We know from Theorem 9.19 that

$T = \sum_k \lambda_k E_{\lambda_k}$; here we have $\lambda_k = \mu_k$ and $E_{\lambda_k}(x) = \langle x, e_k \rangle e_k$ (technically, in Theorem 9.19, E_{λ_k} is the projection onto M_k and we need to sum over all e_k in the basis for M_k to get such an E_{λ_k}, but we ultimately take such a sum, as follows). So $Tx = \sum_k \mu_k \langle x, e_k \rangle e_k$.

Definition. An operator S on a Hilbert space K (so $S : K \to K$) is *unilaterily equivalent* to an operator T on Hilbert space H (so $T : H \to H$) if there is a bijective isometry $U : K \to H$ such that $S = U^{-1}TU$.

Note. We have the mappings:

$$\begin{tikzcd} K \arrow[r, "U"] \arrow[d, "S"'] & H \arrow[d, "T"] \\ K & H \arrow[l, "U^{-1}"'] \end{tikzcd}$$

Theorem 9.20. A compact, self adjoint operator T on a separable Hilbert space is unitarily equivalent to a multiplication operator M_f on ℓ^2.

Note. Since ℓ^2 is a relatively conceptually tangible Hilbert space and multiplication operators are also tangible, then Theorem 9.20 gives a nice way to think about compact, self adjoint operators (on *separable* Hilbert spaces).