9.6. Invariant Subspaces

Note. In this section, we state a conjecture from functional analysis. The conjecture seems to date from the late 1940s, following publications of A. Beurling, or possible from work of von Neumann. For the history of the problem, see B.S. Yadav's "The Invariant Subspace Problem," *Nieuw Archief voor Wiskunde*, Series 5/6, Number 2, June 2005, 148–152 (available from `http://www.nieuwarchief.nl/serie5/pdf/naw5-2005-06-2-148.pdf`, last accessed 7/6/2013). The problem was mentioned in the PBS program *NOVA* in an episode called *A Mathematical Mystery Tour* (Program 1208, first aired March 5, 1985). It was on a list of some of the most important unsolved problems of the day. Called the *Invariant Subspace Problem for Hilbert Spaces*, it is stated as: "Does every bounded operator on a Hilbert space have a nontrivial, closed, invariant [sub]space?"

A screen shot from *A Mathematical Mystery Tour* stating the invariant subspace problem

Recall. For $T \in \mathcal{B}(X)$, a subspace M of X is an invariant subspace for T if $T(M) \subseteq M$.

Note. As Promislow states it, the Invariant Subspace Problem is: Determine whether or not any bounded linear operator on a separable Banach space of dimension bigger than 1 has a nontrivial closed invariant subspace. In 1972, Per Enflo found a Banach space counterexample, which he published in 1987. Other counterexamples were published by Charles Read in 1984 and 1985. The problem remains **unsolved** for Hilbert spaces, however.

Note. The problem was solved in the positive for compact operators on a Hilbert space by Aronszajn and Smith in 1960. This result was simplified and shown for compact operators on Banach spaces by V. I. Lomonosov in 1973 (On Invariant Subspaces of Families of Operators Commuting with a Compact Operator, *Funk. Analiz i ego Prilozh*, **7**, 55–56). Promislow's presentation follows that of J. R. Giles in *Introduction to the Analysis of Normed Linear Spaces*, Cambridge University Press (2000).

Note 1. If T has an eigenvalue then the eigenspace for that eigenvalue is an invariant subspace and is not trivial (unless T is a scalar multiple of I in which case every subspace is invariant).

Note 2. For bounded $T \in \mathcal{B}(X)$, consider $\mathcal{P} = \{p(T) \mid p \text{ is a polynomial}, p(T) \in \mathcal{B}(X)\}$. Let $x \in X$, $x \neq 0$. Define the subset of X, $\mathcal{P}x = \{p(T)(x) \mid p(T) \in \mathcal{P}\}$. Notice that T is bounded and $p(T) \in \mathcal{P}$ is bounded, so $Tp(T)$ is bounded by Theorem 2.8. So $Tp(T) \in \mathcal{P}$ for all $p(T) \in \mathcal{P}$. For any nonzero $x \in X$, $\mathcal{P}x$ is a subspace since linear combinations of polynomials is a polynomial and boundedness of a linear combination follows from the Triangle Inequality and the Scalar Property (see Section 2.2) for the operator norm. Also, $\mathcal{P}x$ is an invariant subspace since $p(T) \in \mathcal{P}$ implies $Tp(T) \in \mathcal{P}$ and so $T(p(T)x) \in \mathcal{P}x$ for all $p(T) \in \mathcal{P}$. Since T is continuous, the closure of this space is a closed invariant subspace. The only possible problem could be that $\mathcal{P}x$ is dense in X for all x, in which case the closure is not a proper subspace.

Theorem 9.21. Let X be a complex Banach space of dimension greater than 1. Any compact $T \in \mathcal{B}(X)$ has a closed proper invariant subspace.

Theorem 9.22. Let X be a complex Banach space of dimension greater than 1. If $A \in \mathcal{B}(X)$ commutes with a nonzero compact operator T, then A has an invariant proper subspace.

Note. So the invariant subspace property does not hold for general bounded linear operators on a Banach space, as shown by Enflo and others. But it does hold for bounded compact operators on Banach spaces and bounded operators which commute with a (nonzero) bounded compact operator on a Banach space.

Since Hilbert spaces are also Banach spaces, then the property holds for bounded compact operators on Hilbert spaces as well (as originally shown by Aronszajn and Smith). The question concerning general bounded linear operators on a Hilbert space is what the "The Invariant Subspace Problem" is.

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