Chapter 5. Vector Spaces, Hilbert Spaces, and the L^2 Space

5.1. Groups, Fields, and Vector Spaces

Note. In this section, we set the stage for exploring Hilbert spaces by reviewing some material from Linear Algebra, including some new ideas about bases.

Definition 5.1.1. A *group* is a set of elements G along with a mapping (called a *binary operation*) $\star : G \times G \to G$ such that

(1) There exists an element $e \in G$ such that for all $g \in G$, $e \star g = g \star e = g$. This element e is called the *identity element* of group G.

(2) For any element $g \in G$ there exists a unique element $h \in G$ such that $g \star h = h \star g = e$. Element h is the *inverse* of g and is denoted $h = g^{-1}$.

(3) For all $g, h, j \in G$, $g \star (h \star j) = (g \star h) \star j$. That is, $\star$ is *associative*.

We denote the group as $\langle G, \star \rangle$. If, in addition, for all $g, h \in G$, $g \star h = h \star g$ then G is an *Abelian* (or *commutative*) group. A *subgroup* of group $\langle G, \star \rangle$ is a subset S of G which is a group under $\star$.

Example 5.1.1. Some additive groups are $\langle \mathbb{Z}_n, + \rangle$. $\langle \mathbb{Z}, + \rangle$, $\langle \mathbb{Q}, + \rangle$, $\langle \mathbb{R}, + \rangle$, $\langle \mathbb{C}, + \rangle$, $\langle \mathbb{R}^n, + \rangle$, and $\langle \mathbb{C}^n, + \rangle$,

Definition 5.1.2. A *field* is a set of elements $\mathbb{F}$ along with two mappings, called *addition*, denoted $+$, and *multiplication*, denoted $\cdot$, where $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, such that $\langle \mathbb{F}, + \rangle$ is an Abelian group with identity element 0 and $\langle \mathbb{F} \setminus \{0\}, \cdot \rangle$ is an Abelian group. The identity element of $\langle \mathbb{F} \setminus \{0\}, \cdot \rangle$ is denoted by 1 and called *unity*. We denote the field as $\langle \mathbb{F}, +, \cdot \rangle$. A *subfield* of $\langle \mathbb{F}, +, \cdot \rangle$ is a subset $\mathbb{S}$ of $\mathbb{F}$ such that $\langle \mathbb{S}, +, \cdot \rangle$ is a field.

Example 5.1.2. Some fields are $\langle \mathbb{Z}_p, +, \cdot \rangle$ where p is prime, $\langle \mathbb{Q}, +, \cdot \rangle$, $\langle \mathbb{R}, +, \cdot \rangle$, and $\langle \mathbb{C}, +, \cdot \rangle$.

Definition 5.1.3. A *vector space* over field $\mathbb{F}$ (the elements of which are called *scalars*) is a set V of elements called *vectors* such that

(a) A mapping called *addition*, denoted $+$, is defined such that $+ : V \times V \to V$ and $\langle V, + \rangle$ is an Abelian group. The identity element of this group is denoted $\mathbf{0}$.

There is a mapping from $\mathbb{F} \times V :\to V$ called *scalar multiplication*. Such that for all $a, b \in \mathbb{F}$ and for all $\mathbf{u}, \mathbf{v} \in V$:

(b) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ (distribution of scalar multiplication over vector addition),

(c) $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$ (distribution of scalar multiplication over scalar addition),

(d) $a(b\mathbf{v}) = (a \cdot b)\mathbf{v}$ (associativity of scalar multiplication),

(e) $1\mathbf{v} = \mathbf{v}$, and

(f) $0\mathbf{v} = \mathbf{0}$.

Example 5.1.3. Some vector spaces are:

(a) $\mathbb{Q}^n = \langle V, \mathbb{Q}\rangle$ where $V = \{(q_1, q_2, \ldots, q_n) \mid q_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n\}$, and scalar multiplication and vector addition are defined componentwise.

(b) $\mathbb{R}^n = \langle V, \mathbb{R}\rangle$ where $V = \{(r_1, r_2, \ldots, r_n) \mid r_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$, and scalar multiplication and vector addition are defined componentwise.

(c) $\mathbb{C}^n = \langle V, \mathbb{C}\rangle$ where $V = \{(c_1, c_2, \ldots, c_n) \mid c_i \in \mathbb{C} \text{ for } 1 \leq i \leq n\}$, and scalar multiplication and vector addition are defined componentwise.

(d) $\mathbb{F}^n = \langle V, \mathbb{F}\rangle$ where $V = \{(f_1, f_2, \ldots, f_n) \mid f_i \in \mathbb{F} \text{ for } 1 \leq i \leq n\}$, and scalar multiplication and vector addition are defined componentwise.

(e) $\ell^2(\mathbb{R}) = \langle V, \mathbb{R}\rangle$ where $V = \left\{(r_1, r_2, r_3, \ldots) \mid r_i \in \mathbb{R} \text{ for } i \geq 1 \text{ and } \displaystyle\sum_{i=1}^{\infty} r_i^2 < \infty\right\}$, and scalar multiplication and vector addition are defined componentwise.

(f) $\ell^2(\mathbb{C}) = \langle V, \mathbb{C}\rangle$ where $V = \left\{(c_1, c_2, c_3, \ldots) \mid c_i \in \mathbb{C} \text{ for } i \geq 1 \text{ and } \displaystyle\sum_{i=1}^{\infty} |c_i|^2 < \infty\right\}$, and scalar multiplication and vector addition are defined componentwise.

Definition 5.1.4. Suppose $\langle V, \mathbb{F}\rangle$ is a vector space. A *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ is a sum of the form $f_1\mathbf{v}_1 + f_2\mathbf{v}_2 + \cdots + f_n\mathbf{v}_n$ where $f_1, f_2, \ldots, f_n \in \mathbb{F}$ are scalars. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is *linearly independent* if $f_1\mathbf{v}_1 + f_2\mathbf{v}_2 + \cdots + f_n\mathbf{v}_n = \mathbf{0}$ only when $f_1 = f_2 = \cdots = f_n = 0$. The *span* of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$ is the set of all linear combinations of the vectors: $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{f_1\mathbf{v}_1 + f_2\mathbf{v}_2 + \cdots + f_n\mathbf{v}_n \mid f_1, f_2, \ldots, f_n \in \mathbb{F}\}$. A *basis* for a vector space is a linearly independent spanning set of the vector space. A vector space is *finite dimensional* if it has a basis of finite cardinality.

Lemma 5.1.1. Consider the homogeneous system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & & & \ddots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

with coefficients a_{ij} ($1 \le i \le m$, $1 \le j \le n$) and unknowns x_k ($1 \le k \le n$) from field $\mathbb{F}$. If $n > m$ then the system has a nontrivial solution (that is, a solution $x_1, x_2, \ldots, x_n$ where $x_k \neq 0$ for some $1 \le k \le n$).

Note. Lemma 5.1.1 is familiar from Linear Algebra where is it proved for $\mathbb{F} = \mathbb{R}$. Notice the proof given in the chapter is for any field $\mathbb{F}$.

Theorem 5.1.1. Let $\langle V, \mathbb{F}\rangle$ be a vector space with bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$. Then $n = m$.

Definition 5.1.5. If vector space $\langle V, \mathbb{F}\rangle$ is a finite dimensional vector space, then the *dimension* of the vector space is the cardinality of a basis.

Definition 5.1.6. Two vector spaces over the same field $\mathbb{F}$, $\langle V, \mathbb{F}\rangle$ and $\langle W, \mathbb{F}\rangle$, are *isomorphic* if there is a one-to-one and onto mapping $\varphi : V \to W$ such that for all $f, f' \in \mathbb{F}$ and $\mathbf{v}, \mathbf{v}' \in V$, we have: $\varphi(f\mathbf{v} + f'\mathbf{v}') = f\varphi(\mathbf{v}) + f'\varphi(\mathbf{v}')$.

Note. We are now prepared to completely classify finite dimensional vector spaces. The following result gives us the answer to the question "What does a finite dimensional vector space *look like*?" More precisely, this result tells us, up to isomorphism, what an n-dimensional vector space is. We raise this result to the status of a "fundamental theorem" and declare it the *Fundamental Theorem of Finite Dimensional Vector Spaces*.

Theorem 5.1.2. The Fundamental Theorem of Finite Dimensional Vector Spaces.
If $\langle V, \mathbb{F}\rangle$ is an n-dimensional vector space, then $\langle V, \mathbb{F}\rangle$ is isomorphic to $\mathbb{F}^n = \langle V^*, \mathbb{F}\rangle$ where $V^* = \{(f_1, f_2, \ldots, f_n) \mid f_1, f_2, \ldots, f_n \in \mathbb{F}\}$, and scalar multiplication and vector addition are defined componentwise.

Note. Now that we know what an n-dimensional vector space "looks like," we use the Fundamental Theorem of Finite Dimensional Vector Spaces to classify certain transformations of these vector spaces.

Definition 5.1.7. A transformation T mapping one vector space $\langle V, \mathbb{F}\rangle$ into another $\langle W, \mathbb{F}\rangle$ is a *linear transformation* if for all $\mathbf{v}, \mathbf{v}' \in V$ and for all $f, f' \in \mathbb{F}$, we have $T(f\mathbf{v} + f'\mathbf{v}') = fT(\mathbf{v}) + f'T(\mathbf{v}')$.

Definition. The *standard basis* of $\mathbb{F}^n$: $\{(1,0,0,\ldots,0),\ (0,1,0,\ldots,0),\ \ldots,\ (0,0,0,\ldots,0,1,0,\ldots,0),\ \ldots,\ (0,0,\ldots,0,1)\}$. We commonly represent the set of standard basis vectors as $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$. If $\mathbf{v}\in\mathbb{F}^n$ and $\mathbf{v}=v_1\mathbf{e}_1+v_2\mathbf{e}_2+\cdots+v_n\mathbf{e}_n$, then we represent $\mathbf{v}$ as $(v_1,v_2,\ldots,v_n)$.

Theorem 5.1.3. If T is a linear transformation from n-dimensional vector space $\langle V,\mathbb{F}\rangle$ to m-dimensional vector space $\langle W,\mathbb{F}\rangle$ then T is equivalent to the action of an $m\times n$ matrix $A_T:\mathbb{F}^n\to\mathbb{F}^m$.

Note. Since finite dimensional vector spaces are totally classified, we now turn our attention to infinite dimensional vector spaces. In this study, we modify some of the above definitions (*basis*, *span*, and *linear combination*). But first, if we keep the above definitions and still define a basis as a linearly independent spanning set (where all linear combinations are finite) then such a basis is called a *Hamel basis*. We will show that every vector space has a Hamel basis, but the argument requires an equivalent of the Axiom of Choice called Zorn's Lemma.

Definition. For set X, any subset of $X\times X$ is a binary relation on X. A relation R on X is *reflexive* if for all $x\in X$, $(x,x)\in R$. R is *symmetric* if $(x,y)\in R$ implies $(y,x)\in R$. R is *transitive* if $(x,y),(y,z)\in R$ implies $(x,z)\in R$.

Definition 1.2.3. A relation R on X is an *equivalence relation* if it is reflexive, symmetric, and transitive. A relation is *antisymmetric* if $(x, y) \in R$ and $(y, x) \in R$ implies that $x = y$. A relation is a *partial ordering* if it is reflexive, antisymmetric and transitive, and is denoted $\leq$. R is a *total ordering* if R is a partial ordering and for any $x, y \in X$, either $x \leq y$ or $y \leq x$.

Definition. If set X is partially ordered by $\leq$, a *maximal* (or *minimal*) element of X is $x \in X$ such that $x \leq y$ (or $y \leq x$) implies $y = x$. If $E \subset X$, an *upper bound* for E is an element $x \in X$ such that $y \leq x$ for all $y \in E$.

Note. The following result is equivalent to the Axiom of Choice.

Zorn's Lemma. If X is a partially ordered set and every totally ordered subset of X has an upper bound, then X has a maximal element.

Theorem 5.1.4. Let $\langle V, \mathbb{F}\rangle$ be a vector space. Then there exists a set of vectors $B \subset V$ such that (1) B is linearly independent and (2) for any $\mathbf{v} \in V$ there exists finite sets $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subset B$ and $\{f_1, f_2, \ldots, f_n\}$ such that $\mathbf{v} = f_1\mathbf{b}_1 + f_2\mathbf{b}_2 + \cdots + f_n\mathbf{b}_n$. That is, B is a Hamel basis for $\langle V, \mathbb{F}\rangle$.

Note. An interesting result concerning Hamel bases for a given vector space is the following:

Exercise 5.1.3. If B_1 and B_2 are Hamel bases for a given infinite dimensional vector space, then B_1 and B_2 are of the same cardinality.

We need two results from set theory. From Hungerford's *Algebra* (1974) page 17:

Theorem 0.8.6. The Schroeder-Bernstein Theorem. If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

From Hungerford's *Algebra* (1974) page 22:

Exercise 0.8.11. If J is an infinite set, and for each $i \in J$ set A_i is a finite set, then $|\cup_{j \in J} A_j| \leq |J|$.

Note. We are new ready for a proof of Exercise 5.1.3.

Note. Since the proof of Theorem 5.1.4 requires Zorn's Lemma, this means that it is not practical to actually FIND a Hamel basis for an infinite dimensional vector space.

Note. We now modify some of the definitions we were using and, generally, replace the idea of "finite linear combination" with "series." But by passing to the infinite requires us to discuss limits of partial sums and hence we need a metric (or at least a topology—see Royden's *Real Analysis*, 3rd Edition, Section 10.5 "Topological Vector Spaces").

Definition 5.1.8. Let $\langle V, \mathbb{F}\rangle$ be a vector space with metric m. Then a countable set of vectors $B \subset V$ is a *Schauder basis* for $\langle V, \mathbb{F}\rangle$ if for each $\mathbf{v} \in V$ there is a unique ordered set of scalars $\{f_1, f_2, \ldots\} \subset \mathbb{F}$ such that $\mathbf{v} = \sum_{n=1}^{\infty} f_n \mathbf{b}_n$. That is, $\lim_{n\to\infty} m\left(\mathbf{v}, \sum_{i=1}^{n} f_i \mathbf{b}_i\right) = 0$.

Note. The uniqueness requirement insures the set B is "linearly independent" in the sense that $\mathbf{0} = \sum_{n=1}^{\infty} f_n \mathbf{b}_n$ if and only if $f_n = 0$ for all $n \in \mathbb{N}$.

Note. Juliusz P. Schauder was born in what is today the Ukraine, but at the time was Poland. He was drafted into the Austro-Hungarian army after finishing high school in 1917. He was taken prisoner in Italy. After the first world war he entered Jan Kasimerz University and earned his doctorate in 1923. Others had studied bases of infinite dimensional spaces, but Schauder gave his definition in "Zur Theorie stetiger Abbildungen in Funktionalraumen" (On the Theory of Continuous Maps in Functional Spaces), *Mathematische Zeitschrift*, **26** (1927), 47-65. Schauder studied topology and functional analysis. He was Jewish and died at the hands of the Nazis in 1943. This information (and the photo below) are from the MacTutor History of Mathematics Archive biography of Schauder.

Juliusz P. Schauder (1899–1943)