Note. In this section, we introduce an inner product on a vector space. This will allow us to bring much of the geometry of $\mathbb{R}^n$ into the infinite dimensional setting.

**Definition 5.2.1.** A vector space with complex scalars $\langle V, \mathbb{C} \rangle$ is an *inner product space* (also called a *Euclidean Space* or a *Pre-Hilbert Space*) if there is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $u, v, w \in V$ and $a \in \mathbb{C}$ we have:

(a) $\langle v, v \rangle \in \mathbb{R}$ and $\langle v, v \rangle \geq 0$ with $\langle v, v \rangle = 0$ if and only if $v = 0$,

(b) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,

(c) $\langle u, av \rangle = a \langle u, v \rangle$, and

(d) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ where the overline represents the operation of complex conjugation.

The function $\langle \cdot, \cdot \rangle$ is called an *inner product*. 
Notice that properties (b), (c), and (d) of Definition 5.2.1 combine to imply that

\[ \langle u, av + bw \rangle = a\langle u, v \rangle + b\langle u, w \rangle \]

and

\[ \langle au + bv, w \rangle = \overline{a}\langle u, w \rangle + \overline{b}\langle u, w \rangle \]

for all relevant vectors and scalars. That is, \( \langle \cdot, \cdot \rangle \) is linear in the second positions and “conjugate-linear” in the first position.

**Note.** We can also define an inner product on a vector space with real scalars by requiring that \( \langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R} \) and by replacing property (d) in Definition 5.2.1 with the requirement that the inner product is symmetric: \( \langle u, v \rangle = \langle v, u \rangle \). Then \( \mathbb{R}^n \) with the usual dot product is an example of a real inner product space.

**Example 5.2.1.** The vector space \( \mathbb{C}^n \) is an inner product space with the inner product defined for \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \) as \( \langle u, v \rangle = \sum_{j=1}^{n} \overline{u_j}v_j \).

**Definition 5.2.2.** For inner product space \( \langle V, \mathbb{C} \rangle \) with inner product \( \langle \cdot, \cdot \rangle \), define the norm induced by the inner product as \( \|v\| = \langle v, v \rangle^{1/2} \) for all \( v \in V \).

**Note.** For all \( a \in \mathbb{F} \) and vectors \( v \) we have that \( \|av\| = |a|\|v\| \).
5.2. Inner Product Spaces

Theorem 5.2.1. Schwarz’s Inequality.
For all \( u, v \) in inner product space \( \langle V, \mathbb{C} \rangle \), we have

\[
|\langle u, v \rangle| \leq \|u\|\|v\|.
\]

Note. As in Linear Algebra, we use the Schwarz Inequality to prove that \( \| \cdot \| \) satisfies the Triangle Inequality.

Theorem 5.2.2. The Triangle Inequality.
For all \( u, v \) in an inner product space \( \langle V, \mathbb{C} \rangle \) we have \( \|u + v\| \leq \|u\| + \|v\| \).

Note. We now see that \( \| \cdot \| \) in fact does satisfy the definition of a norm.

Note. Schematically we have:

\[
\text{(vector spaces)} \supset \text{(normed vector spaces)} \supset \text{(inner product spaces)}.
\]

Definition 5.2.3. Two vectors \( u, v \) in an inner product space are orthogonal if \( \langle u, v \rangle = 0 \). A set of vectors \( \{v_1, v_2, \ldots \} \) is orthogonal if \( \langle v_i, v_j \rangle = 0 \) for \( i \neq j \). This orthogonal set of vectors is orthonormal if in addition \( \langle v_i, v_i \rangle = |v_i|^2 = 1 \) for all \( i \) and, in this case, the vectors are said to be normalized.
**Theorem 5.2.3. The Pythagorean Theorem.**

Let \( \{v_1, v_2, \ldots, v_n\} \) be an orthonormal set of vectors in an inner product space \( \langle V, \mathbb{C} \rangle \). Then for all \( u \in V \)

\[
\|u\|^2 = \sum_{j=1}^{n} |\langle u, v_j \rangle|^2 + \left\| u - \sum_{j=1}^{n} \langle v_j, u \rangle v_j \right\|^2.
\]

**Note.** If we have \( v \) and \( w \) orthogonal and set \( u = v + w \) then the Pythagorean Theorem implies the familiar result that \( \|u\|^2 = \|v\|^2 + \|w\|^2 \).

**Note.** Since the Pythagorean Theorem holds in inner product spaces, then these spaces must be Euclidean (“flat”). This is because the metric induced by the inner product is the Euclidean metric.

**Corollary 5.2.1. Bessel’s Inequality.**

Let \( \{v_1, v_2, \ldots, v_n\} \) be an orthonormal set in an inner product space \( \langle V, \mathbb{C} \rangle \). Then for all \( u \in V \) we have

\[
\|u\|^2 \geq \sum_{j=1}^{n} |\langle u, v_j \rangle|^2.
\]

**Note.** Explicitly, we have the metric \( d : V \times V \to \mathbb{R} \) defined as \( d(u, v) = \|u - v\| \). therefore we can define *Cauchy sequences* and *limits* in an inner product space.
Definition 5.2.5. An inner product space is *complete* if Cauchy sequences converge.

Definition 5.2.6. A complete inner product space is a *Hilbert space*.

Note. Hilbert spaces are special cases of Banach spaces.

Example 5.2.2. Since $\mathbb{R}$ and $\mathbb{C}$ are complete, then $\mathbb{R}^n$ and $\mathbb{C}^n$ are examples of Hilbert spaces (with the familiar dot product as the inner product on $\mathbb{R}$ and the inner product on $\mathbb{C}^n$ as defined in Example 5.2.1. However, we are quite familiar with the structure of $\mathbb{R}^n$ and $\mathbb{C}^n$ from our studies of linear algebra. In fact, every real vector space of dimension $n$ is isomorphic to $\mathbb{R}^n$ and every complex vector space of dimension $n$ is isomorphic to $\mathbb{C}^n$ (by the Fundamental Theorem of Finite Dimensional Vector Spaces). So we will next turn our attention to infinite dimensional vector spaces which are also Hilbert spaces.

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