5.2. Inner Product Spaces

Note. In this section, we introduce an inner product on a vector space. This will allow us to bring much of the geometry of $\mathbb{R}^n$ into the infinite dimensional setting.

Definition 5.2.1. A vector space with complex scalars $\langle V, \mathbb{C}\rangle$ is an *inner product space* (also called a *Euclidean Space* or a *Pre-Hilbert Space*) if there is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $a \in \mathbb{C}$ we have:

(a) $\langle \mathbf{v}, \mathbf{v}\rangle \in \mathbb{R}$ and $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ with $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$,

(b) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$,

(c) $\langle \mathbf{u}, a\mathbf{v}\rangle = a\langle \mathbf{u}, \mathbf{v}\rangle$, and

(d) $\langle \mathbf{u}, \mathbf{v}\rangle = \overline{\langle \mathbf{v}, \mathbf{u}\rangle}$ where the overline represents the operation of complex conjugation.

The function $\langle \cdot, \cdot \rangle$ is called an *inner product.*

Note. Notice that properties (b), (c), and (d) of Definition 5.2.1 combine to imply that

$$\langle \mathbf{u}, a\mathbf{v} + b\mathbf{w}\rangle = a\langle \mathbf{u}, \mathbf{v}\rangle + b\langle \mathbf{u}, \mathbf{w}\rangle$$

and

$$\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w}\rangle = \overline{a}\langle \mathbf{u}, \mathbf{w}\rangle + \overline{b}\langle \mathbf{v}, \mathbf{w}\rangle$$

for all relevant vectors and scalars. That is, $\langle \cdot, \cdot \rangle$ is linear in the second positions and "conjugate-linear" in the first position.

Note. We can also define an inner product on a vector space with real scalars by requiring that $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ and by replacing property (d) in Definition 5.2.1 with the requirement that the inner product is symmetric: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$. Then $\mathbb{R}^n$ with the usual dot product is an example of a real inner product space.

Example 5.2.1. The vector space $\mathbb{C}^n$ is an inner product space with the inner product defined for $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ as $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{j=1}^{n} \overline{u_j} v_j$.

Definition 5.2.2. For inner product space $\langle V, \mathbb{C} \rangle$ with inner product $\langle \cdot, \cdot \rangle$, define the *norm* induced by the inner product as $\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$ for all $\mathbf{v} \in V$.

Note. For all $a \in \mathbb{F}$ and vectors $\mathbf{v}$ we have that $\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$.

Theorem 5.2.1. Schwarz's Inequality.

For all $\mathbf{u}, \mathbf{v}$ in inner product space $\langle V, \mathbb{C} \rangle$, we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Note. As in Linear Algebra, we use the Schwarz Inequality to prove that $\| \cdot \|$ satisfies the Triangle Inequality.

Theorem 5.2.2. The Triangle Inequality.

For all $\mathbf{u}, \mathbf{v}$ in an inner product space $\langle V, \mathbb{C}\rangle$ we have $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

Note. We now see that $\|\cdot\|$ in fact does satisfy the definition of a norm.

Note. Schematically we have:

$$(\text{vector spaces}) \supset (\text{normed vector spaces}) \supset (\text{inner product spaces}).$$

Definition 5.2.3. Two vectors $\mathbf{u}, \mathbf{v}$ in an inner product space are *orthogonal* if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ is orthogonal if $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = 0$ for $i \neq j$. This orthogonal set of vectors is *orthonormal* if in addition $\langle \mathbf{v}_i, \mathbf{v}_i\rangle = \|\mathbf{v}_i\|^2 = 1$ for all i and, in this case, the vectors are said to be *normalized*.

Theorem 5.2.3. The Pythagorean Theorem.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal set of vectors in an inner product space $\langle V, \mathbb{C}\rangle$. Then for all $\mathbf{u} \in V$

$$\|\mathbf{u}\|^2 = \sum_{j=1}^{n} |\langle \mathbf{u}, \mathbf{v}_j\rangle|^2 + \left\| \mathbf{u} - \sum_{j=1}^{n} \langle \mathbf{v}_j, \mathbf{u}\rangle \mathbf{v}_j \right\|^2.$$

Note. If we have $\mathbf{v}$ and $\mathbf{w}$ orthogonal and set $\mathbf{u} = \mathbf{v} + \mathbf{w}$ then the Pythagorean Theorem implies the familiar result that $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

Note. Since the Pythagorean Theorem holds in inner product spaces, then these spaces must be Euclidean ("flat"). This is because the metric induced by the inner product is the Euclidean metric.

Corollary 5.2.1. Bessel's Inequality.
Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal set in an inner product space $\langle V, \mathbb{C}\rangle$. Then for all $\mathbf{u} \in V$ we have

$$\|\mathbf{u}\|^2 \geq \sum_{j=1}^{n} |\langle \mathbf{u}, \mathbf{v}_j \rangle|^2.$$

Note. Explicitly, we have the metric $d : V \times V \to \mathbb{R}$ defined as $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$. Therefore we can define *Cauchy sequences* and *limits* in an inner product space.

Definition 5.2.5. An inner product space is *complete* if Cauchy sequences converge.

Definition 5.2.6. A complete inner product space is a *Hilbert space.*

Note. Hilbert spaces are special cases of Banach spaces.

Example 5.2.2. Since $\mathbb{R}$ and $\mathbb{C}$ are complete, then $\mathbb{R}^n$ and $\mathbb{C}^n$ are examples of Hilbert spaces (with the familiar dot product as the inner product on $\mathbb{R}$ and the inner product on $\mathbb{C}^n$ as defined in Example 5.2.1). However, we are quite familiar with the structure of $\mathbb{R}^n$ and $\mathbb{C}^n$ from our studies of linear algebra. In fact, every real vector space of dimension n is isomorphic to $\mathbb{R}^n$ and every complex vector space of dimension n is isomorphic to $\mathbb{C}^n$ (by the Fundamental Theorem of Finite Dimensional Vector Spaces). So we will next turn our attention to infinite dimensional vector spaces which are also Hilbert spaces.

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