5.3. The Space $L^2$

**Note.** In this brief section, we introduce a fundamental inner product space which will play a huge role in the theory of Hilbert spaces.

**Definition.** For interval $[a, b] \subset \mathbb{R}$, we define the space $L^2([a, b])$ as

$$L^2([a, b]) = \left\{ f \left| \int_{[a,b]} |f|^2 < \infty \right. \right\}.$$

**Note.** In our exploration of Banach spaces and $L^p$ spaces, we have seen that $L^2([a, b])$ is a complete space with respect to the $L^2$ norm. This is done in Promislow’s *A First Course in Functional Analysis* in Section 2.9, and in Royden and Fitzpatrick’s *Real Analysis*, 4th Edition, in Chapter 7. In fact, Royden and Fitzpatrick’s 4th Edition (unlike the previous editions) deal with $L^p$ spaces over general measurable sets, instead of over just intervals.

**Note.** An inner product on $L^2([a, b])$ is given by $\langle f, g \rangle = \int_{[a,b]} fg$ where $f$ and $g$ are real valued. In the event that $f$ and $g$ are functions of a complex variable, we have that $\langle f, g \rangle = \int_{[a,b]} f \overline{g}$. These inner products induce the $L^2$ norm, so $L^2([a, b])$ is a a complete inner product space and therefore a Hilbert space. We will see in the next section that, in some sense, all (infinite dimensional) Hilbert spaces are the same as $L^2$. 
Note. Examples of sequence spaces which are Hilbert spaces are

\[ \ell^2(\mathbb{R}) = \left\{ (x_1, x_2, \cdots) \mid x_n \in \mathbb{R}, \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \]

and

\[ \ell^2(\mathbb{C}) = \left\{ (z_1, z_2, \cdots) \mid z_n \in \mathbb{R}, \sum_{n=1}^{\infty} |z_n|^2 < \infty \right\} . \]