5.3. The Space L^2

Note. In this brief section, we introduce a fundamental inner product space which will play a huge role in the theory of Hilbert spaces.

Definition. For interval $[a,b] \subset \mathbb{R}$, we define the space $L^2([a,b])$ as

$$L^2([a,b]) = \left\{ f \,\middle|\, \int_{[a,b]} |f|^2 < \infty \right\}.$$

Note. In our exploration of Banach spaces and L^p spaces, we have seen that $L^2([a,b])$ is a complete space with respect to the L^2 norm. This is done in Promislow's *A First Course in Functional Analysis* (see my online notes for Fundamentals of Functional Analysis [MATH 5740] on Section 2.9. L^p Spaces), and in Royden and Fitzpatrick's *Real Analysis*, 4th Edition, in "Chapter 7. The L^p Spaces: Completeness and Approximation" (see my online notes for Real Analysis 2 [MATH 5220]). In fact, Royden and Fitzpatrick's 4th Edition (unlike the previous editions) deals with L^p spaces over general measurable sets, instead of over just intervals.

Note. An inner product on $L^2([a,b])$ is given by $\langle f, g\rangle = \int_{[a,b]} fg$ where f and g are real valued. In the event that f and g are complex valued functions of a real variable, we have that $\langle f, g\rangle = \int_{[a,b]} \overline{f}g$. These inner products induce the L^2 norm, so $L^2([a,b])$ is a complete inner product space and therefore a Hilbert space. We will see in the next section that, in some sense, all (infinite dimensional) Hilbert spaces are the same as L^2.

Note. Examples of sequence spaces which are Hilbert spaces are

$$\ell^2(\mathbb{R}) = \left\{ (x_1, x_2, \cdots) \mid x_n \in \mathbb{R}, \sum_{n=1}^{\infty} x_n^2 < \infty \right\}$$

and

$$\ell^2(\mathbb{C}) = \left\{ (z_1, z_2, \cdots) \mid z_n \in \mathbb{C}, \sum_{n=1}^{\infty} |z_n|^2 < \infty \right\}.$$

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