5.4. Projections and Hilbert Space Isomorphisms

Note. In this section, we drag many of the geometric properties of $\mathbb{R}^n$ into $\ell^2$ and show that every infinite dimensional Hilbert space is isomorphic to $\ell^2$.

Definition. Therefore, in an inner product space we define the projection of $f$ onto nonzero $g$ in a Hilbert space as $\text{proj}_g(f) = \frac{\langle f, g \rangle}{\langle g, g \rangle} g$. For a nonempty set $S$ in a Hilbert space $H$, we say that $h \in H$ is orthogonal to $S$ if $\langle h, s \rangle = 0$ for all $s \in S$. The orthogonal complement of $S$ is

$$S^\perp = \{ h \in H \mid \langle h, s \rangle = 0 \text{ for all } s \in S \}.$$ 

($S^\perp$ is pronounced “$S$ perp” and $S^\perp$ is sometimes called the “perp space” of $S$.) In fact, $S^\perp$ is itself a Hilbert space:

**Theorem 5.4.1.** For any nonempty set $S$ in a Hilbert space $H$, the set $S^\perp$ is a Hilbert space.

Note. We now use the idea of an orthogonal complement to decompose a Hilbert space into subspaces. As we will see, the decomposition is an algebraic and not a set theoretic decomposition (it will involve vector addition as opposed to set union).

**Theorem 5.4.2.** Let $S$ be a subspace of a Hilbert space $H$ (that is, the set of vectors in $S$ is a subset of the set of vectors in $H$ and $S$ itself is a Hilbert space). Then for any $h \in H$, there exists a unique $t \in S$ such that $\inf_{s \in S} \| h - s \| = \| h - t \|$. 
5.4. Projections and Hilbert Space Isomorphisms

Note. We now use Theorem 5.4.2 to uniquely decompose elements of $H$ into a sum of an element of $S$ and an element of $S^\perp$.

**Theorem 5.4.3.** Let $S$ be a subspace of a Hilbert space $H$. Then for all $h \in H$, there exists a unique decomposition of the form $h = s + s'$ where $s \in S$ and $s' \in S^\perp$.

**Definition.** For $h \in H$, a Hilbert space, the vector $s \in S$ described in Theorem 5.4.3 is the projection of vector $h \in H$ onto subspace $S$, denoted $\text{proj}_S(h) = s$. We also say that $H$ can be written as the direct sum of $S$ and $S^\perp$, denoted $H = S \oplus S^\perp$, and that $H$ has this as an (orthogonal) decomposition.

**Definition.** In a Hilbert space, a set of nonzero vectors is orthogonal if the vectors are pairwise orthogonal and if, in addition, each vector is a unit vector then the set is an orthonormal set.

**Definition 5.4.1.** A Schauder basis of a Hilbert space which is also an orthonormal set is called an orthonormal basis or a Riesz basis.

Note. The following result is a simple consequence of the Gram-Schmidt process.

**Theorem 5.4.4.** A Hilbert space with a Schauder basis has an orthonormal basis.
Theorem 5.4.5. If $R = \{ r_1, r_2, \ldots \}$ is an orthonormal basis for a Hilbert space $H$ and if $h \in H$, then 
\[ h = \sum_{k=1}^{\infty} \langle h, r_k \rangle r_k. \]

Theorem 5.4.6. If $R = \{ r_1, r_2, \ldots \}$ is an orthonormal basis for a Hilbert space $H$, let $R_k = \text{span}\{ r_1, r_2, \ldots, r_{k-1} \}$ and let $h \in H$. Then $\inf_{s \in R_k} \| h - s \| = \| h - t \|$ where $t = \sum_{n=1}^{k-1} \langle h, r_n \rangle r_n$. That is, the best approximation of $h$ is given by partial sums of the orthonormal series of $h$ (i.e., $t = \text{proj}_{R_k}(h)$).

Theorem 5.4.7. If $R = \{ r_1, r_2, \ldots \}$ is an orthonormal basis for a Hilbert space $H$ and for $h \in H$ we have $h = \sum_{k=1}^{\infty} \langle h, r_k \rangle r_k$, then $\| h \|^2 = \sum_{k=1}^{\infty} |a_k|^2$ where $a_k = \langle h, r_k \rangle$.

Definition 5.4.2. A Hilbert space with a countable dense subset is separable. That is, a separable Hilbert space $H$ has a subset $D = \{ d_1, d_2, \ldots \}$ such that for any $h \in H$ and for all $\varepsilon > 0$, there exists $d_k \in D$ with $\| h - d_k \| < \varepsilon$. Therefore the (topological) closure of $D$ is $H$.

Note. Many texts study separable Hilbert spaces. We are interested in Hilbert spaces with Schauder bases. The following shows that these are equivalent.

Theorem 5.4.8. A Hilbert space with scalar field $\mathbb{R}$ or $\mathbb{C}$ is separable if and only if it has a countable orthonormal basis.
**Definition 5.4.3.** Let $H_1$ and $H_2$ be Hilbert spaces. If there exists a one-to-one and onto linear mapping $\pi : H_1 \to H_2$ such that inner products are preserved: $$\langle h, h' \rangle = \langle \pi(h), \pi(h') \rangle$$ for all $h, h' \in H_1$, then $\pi$ is a *Hilbert space isomorphism* and $H_1$ and $H_2$ are *isomorphic*.

**Note.** We are now ready to extend the Fundamental Theorem of Finite Dimensional Vector Spaces to the infinite dimensional case.

**Theorem 5.4.9. Fundamental Theorem of Infinite Dimensional Vector Spaces.**

Let $H$ be a Hilbert space with a countable infinite orthonormal basis. Then $H$ is isomorphic to $\ell^2$.

**Note.** The Fundamental Theorem of Infinite Dimensional Vector Spaces states that all Hilbert spaces with a countable infinite orthonormal basis are isomorphic, since they are all isomorphic to $\ell^2$. So the answer to the big question “What does an infinite dimensional vector space look like?,” is “$\ell^2$!”

**Note.** Finally, we show that linear transformations from one infinite dimensional Hilbert space to another are represented by matrices (well, infinite matrices) just like linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. 
Theorem 5.4.10. If $T : H_1 \rightarrow H_2$ is a linear transformation where $H_1$ and $H_2$ are Hilbert spaces (over the same field) with countable infinite bases, then $T$ is equivalent to the action of an infinite matrix $(A_{ij})_{i,j \in \mathbb{N}}$.

Note. So we now see that many of the properties of $\mathbb{R}^n$ carry over to the infinite dimensional space $\ell^2$. Though we don’t usually speak of vectors in Hilbert spaces as having magnitude and direction, there is still some validity to this idea. Just as there are $n$ “fundamental directions” $(1,0,0,\ldots,0)$, $(0,1,0,\ldots,0)$, $\ldots$, $(0,0,0,\ldots,0,1)$ in $\mathbb{R}^n$ (the “fundamental” property being given by the fact that every “direction” [i.e., nonzero vector] is a linear combination of these “directions” [i.e., vectors]), there are a countable number of “fundamental directions” in a Hilbert space with an orthonormal basis. This is a particularly tangible idea when we consider the Hilbert space

$$\ell^2 = \left\{ (a_1, a_2, \ldots) \left| \sum_{k=1}^{\infty} |a_k|^2 < \infty, a_k \in \mathbb{R} \right. \right\}$$

with orthonormal basis $R = \{(1,0,0,\ldots),(0,1,0,\ldots),(0,0,1,0,\ldots),\ldots\}$.

Note. Consider the set $R = \{(1,0,0,\ldots),(0,1,0,\ldots),(0,0,1,0,\ldots),\ldots\}$ as a subset of $\ell^2$. This set is closed (since any two elements of $R$ are a distance $\sqrt{2}$ apart and so $R$ consists of isolated points) and bounded (each element is distance 1 from the origin [i.e., the vector $0$]). However, if we take the open covering of $R$ with balls centered on the elements of $R$ with radius $1/2$, then we see that there is no subcover. Therefore $R$ is not compact. So we have violated the Heine-Borel Theorem (well, Heine-Borel only claims to hold in finite dimensions)! In
addition, $R$ is an infinite bounded set without a limit point (in apparent violation of Weierstrass’s Theorem). In the proof of Weierstrass’s Theorem, the finite set is cut in half a countable number of times to produce a limit point. However, in an infinite dimensional space there are so many “directions” that we can create the set $R$ which is infinite and bounded, but the points do not cluster because we have taken advantage of the many directions.

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