5.4. Projections and Hilbert Space Isomorphisms

Note. In this section, we drag many of the geometric properties of $\mathbb{R}^n$ into ℓ^2 and show that every infinite dimensional Hilbert space (with some additional restrictions) is isomorphic to ℓ^2.

Definition. In an inner product space we define the *projection* of f onto nonzero g in a Hilbert space as $\text{proj}_g(f) = \dfrac{\langle f, g\rangle}{\langle g, g\rangle} g$. For a nonempty set S in a Hilbert space H, we say that $h \in H$ is *orthogonal* to S if $\langle h, s\rangle = 0$ for all $s \in S$. The *orthogonal complement* of S is

$$S^\perp = \{h \in H \mid \langle h, s\rangle = 0 \text{ for all } s \in S\}.$$

($S^\perp$ is pronounced "S perp" and $S^\perp$ is sometimes called the "perp space" of S.) In fact, $S^\perp$ is itself a Hilbert space:

Theorem 5.4.1. For any nonempty set S in a Hilbert space H, the set $S^\perp$ is a Hilbert space.

Note. We now use the idea of an orthogonal complement to decompose a Hilbert space into subspaces. As we will see, the decomposition is an algebraic and not a set theoretic decomposition (it will involve vector addition as opposed to set union).

Theorem 5.4.2. Let S be a subspace of a Hilbert space H (that is, the set of vectors in S is a subset of the set of vectors in H and S itself is a Hilbert space). Then for any $h \in H$, there exists a unique $t \in S$ such that $\inf_{s\in S} \|h - s\| = \|h - t\|$.

Note. We now use Theorem 5.4.2 to uniquely decompose elements of H into a sum of an element of S and an element of $S^\perp$.

Theorem 5.4.3. Let S be a subspace of a Hilbert space H. Then for all $h \in H$, there exists a unique decomposition of the form $h = s + s'$ where $s \in S$ and $s' \in S^\perp$.

Definition. For $h \in H$, a Hilbert space, the vector $s \in S$ described in Theorem 5.4.3 is the *projection* of vector $h \in H$ onto subspace S, denoted $\mathrm{proj}_S(h) = s$. We also say that H can be written as the *direct sum* of S and $S^\perp$, denoted $H = S \oplus S^\perp$, and that H has this as an (orthogonal) decomposition.

Definition. In a Hilbert space, a set of nonzero vectors is *orthogonal* if the vectors are pairwise orthogonal and if, in addition, each vector is a unit vector then the set is an *orthonormal* set.

Definition 5.4.1. A Schauder basis of a Hilbert space which is also an orthonormal set is called an *orthonormal basis* or a *Riesz basis*.

Note. The following result is a simple consequence of the Gram-Schmidt process.

Theorem 5.4.4. A Hilbert space with a Schauder basis has an orthonormal basis.

Theorem 5.4.5. If $R = \{r_1, r_2, \ldots\}$ is an orthonormal basis for a Hilbert space H and if $h \in H$, then

$$h = \sum_{k=1}^{\infty} \langle h, r_k \rangle r_k.$$

Theorem 5.4.6. If $R = \{r_1, r_2, \ldots\}$ is an orthonormal basis for a Hilbert space H, let $R_k = \text{span}\{r_1, r_2, \ldots, r_{k-1}\}$ and let $h \in H$. Then $\inf_{s \in R_k} \|h - s\| = \|h - t\|$ where $t = \sum_{n=1}^{k-1} \langle h, r_n \rangle r_n$. That is, the best approximation of h is given by partial sums of the orthonormal series of h (i.e., $t = \text{proj}_{R_k}(h)$).

Note. The next result follows easily from the fact that the inner product is continuous (by Exercise 5.2.6) and the definition of orthonormal.

Theorem 5.4.7. If $R = \{r_1, r_2, \ldots\}$ is an orthonormal basis for a Hilbert space H and for $h \in H$ we have $h = \sum_{k=1}^{\infty} \langle h, r_k \rangle r_k$, then $\|h\|^2 = \sum_{k=1}^{\infty} |a_k|^2$ where $a_k = \langle h, r_k \rangle$.

Definition 5.4.2. A Hilbert space with a countable dense subset is *separable*. That is, a separable Hilbert space H has a subset $D = \{d_1, d_2, \ldots\}$ such that for any $h \in H$ and for all $\varepsilon > 0$, there exists $d_k \in D$ with $\|h - d_k\| < \varepsilon$. Therefore the (topological) closure of D is H.

Note. Many texts study separable Hilbert spaces. We are interested in Hilbert spaces with Schauder bases. The following shows that these are equivalent.

Theorem 5.4.8. A Hilbert space with scalar field $\mathbb{R}$ or $\mathbb{C}$ is separable if and only if it has a countable orthonormal basis.

Definition 5.4.3. Let H_1 and H_2 be Hilbert spaces. If there exists a one-to-one and onto linear mapping $\pi : H_1 \to H_2$ such that inner products are preserved: $\langle h, h' \rangle = \langle \pi(h), \pi(h') \rangle$ for all $h, h' \in H_1$, then π is a *Hilbert space isomorphism* and H_1 and H_2 are *isomorphic*.

Note. We are now ready to extend the Fundamental Theorem of Finite Dimensional Vector Spaces to the infinite dimensional case.

Theorem 5.4.9. Fundamental Theorem of Infinite Dimensional Vector Spaces.

Let H be a Hilbert space with a countable infinite orthonormal basis. Then H is isomorphic to ℓ^2.

Note. The Fundamental Theorem of Infinite Dimensional Vector Spaces states that all Hilbert spaces with a countable infinite orthonormal basis are isomorphic, since they are all isomorphic to ℓ^2. So the answer to the big question "What does an infinite dimensional vector space look like?," is "ℓ^2!"

Note. Finally, we show that linear transformations from one infinite dimensional Hilbert space to another are represented by matrices (well, infinite matrices) just like linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

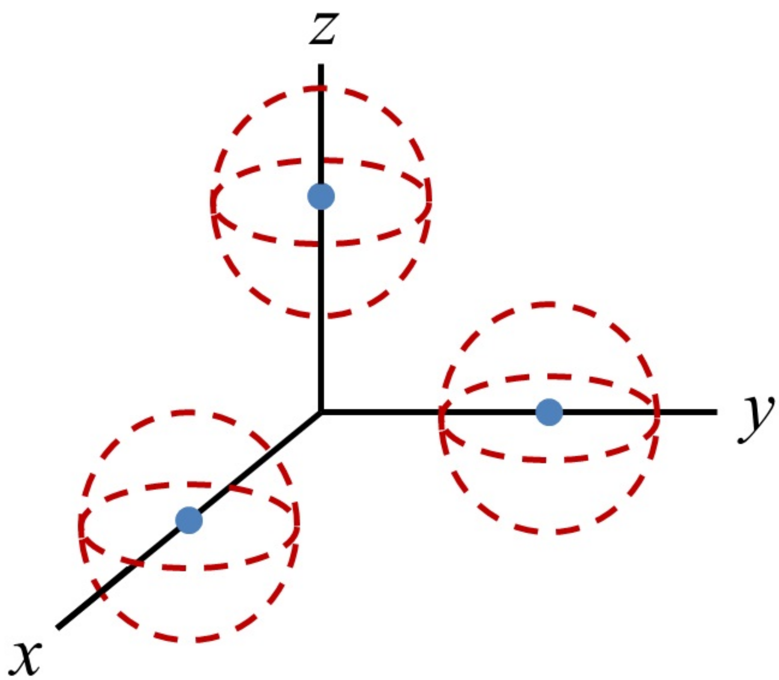
Theorem 5.4.10. If $T : H_1 \to H_2$ is a linear transformation where H_1 and H_2 are Hilbert spaces (over the same field) with countable infinite bases, then T is equivalent to the action of an infinite matrix $(A_{ij})_{i,j\in\mathbb{N}}$.

Note 5.4.A. So we now see that many of the properties of $\mathbb{R}^n$ carry over to the infinite dimensional space ℓ^2. Though we don't usually speak of vectors in Hilbert spaces as having magnitude and direction, there is still some validity to this idea. Just as there are n "fundamental directions" $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots,$ $(0, 0, 0, \ldots, 0, 1)$ in $\mathbb{R}^n$ (the "fundamental" property being given by the fact that every "direction" [i.e., nonzero vector] is a linear combination of these "directions" [i.e., basis vectors]), there are a countable number of "fundamental directions" in a Hilbert space with an orthonormal basis. This is a particularly tangible idea when we consider the Hilbert space

$$\ell^2 = \left\{ (a_1, a_2, \ldots) \;\middle|\; \sum_{k=1}^{\infty} |a_k|^2 < \infty, a_k \in \mathbb{R} \right\}$$

with orthonormal basis $R = \{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots\}$.

Note 5.4.B. Consider the set $R = \{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots\}$ as a subset of ℓ^2. This set is closed (since any two elements of R are a distance $\sqrt{2}$ apart and so R consists of isolated points) and bounded (each element is distance 1 from the origin [i.e., the vector $\mathbf{0}$]). However, if we take the open covering of R with balls centered on the elements of R with radius $1/2$, then we see that there is no subcover. Therefore R is not compact. So we have violated the Heine-Borel Theorem (well, Heine-Borel only claims to hold in finite dimensions)! See the illustration below. In addition, R is an infinite bounded set without a limit point (in apparent violation of Weierstrass's Theorem). In the proof of Weierstrass's Theorem, the finite set is cut in half a countable number of times to produce a limit point. However, in an infinite dimensional space there are so many "directions" that we can create the set R which is infinite and bounded, but the points do not cluster because we have taken advantage of the many directions.



A closed and bounded set in ℓ^2 that is not compact;

only the first three axes of the countably infinite number of axes is shown.